One-dimensional discontinuous piecewise-linear maps

and the dynamics of financial markets[*]

Fabio Tramontana[**] and Frank Westerhoff[***]

[**] University of Pavia, Department of Economics and Quantitative Methods

[***] University of Bamberg, Department of Economics

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1 Introduction

Over the last 15 years, major stock markets around the world have behaved rather turbulently and unpleasantly. Consider, for instance, the evolution of the FTSE MIB, the Italian stock market index, and the DAX, the German stock market index, between 1998 and 2010, as depicted in the top panels of Figures 1 and 2, respectively. First we saw the emergence of the so-called dot-com bubble with stock market peaks around March 2000, followed by a dramatic crash. Afterwards, however, the stock markets recovered, again reaching (almost comparable) highs in 2007, only to crash once again. The crash following autumn 2007 was extremely severe since it triggered, together with other financial market meltdowns, a global economic crisis. The second panels of Figures 1 and 2 show the returns of the two stock markets (defined as log price changes). What is immediately apparent is that these markets are highly volatile, also with respect to their daily price variability. The magnitude of the most extreme price fluctuations as well as the overall level of volatility is simply stunning.

***** Figure 1 and Figure 2 about here *****

Given the negative impact such stock market dynamics may have for the real economy, it is important to understand what drives these markets. Agent-based financial market models have been exploring this important issue for a number of years now (for surveys, see Chiarella et al. 2009, Hommes and Wagener 2009, Lux 2009 and Westerhoff 2009). These models study interactions between heterogeneous market participants who rely on simple technical and fundamental trading rules to determine their orders. It should be noted that the key building blocks of these models are supported by empirical evidence. Most importantly, there are numerous survey studies (see the review of Menkhoff and Taylor 2007) and laboratory experiments (see the review of Hommes 2011) which clearly confirm that financial market participants do indeed rely on trend-extrapolating and mean-reverting trading strategies.

Some agent-based financial-market models can be studied analytically. These models are usually deterministic, and reveal that nonlinear trading rules, switching between (linear) trading

rules and/or market interactions, may lead to irregular endogenous price dynamics. Contributions in this direction include Day and Huang (1990), Kirman (1991), de Grauwe et al. (1993), Lux (1995), Brock and Hommes (1998), Chiarella et al. (2002) and Westerhoff (2004).

Analytically tractable models are usually represented by smooth dynamical systems. However, there are also a few examples where the dynamical system is discontinuous (e.g. Huang and Day 1993, Huang et al. 2010, Tramontana et al. 2010, 2011a). One advantage of these models is that they allow a deeper analytical understanding of the underlying dynamical system. Another advantage is that discontinuous maps also offer interesting and sometimes quite peculiar bifurcation phenomena, enriching our understanding of what may be going on in financial markets. For instance, in some of these models fixed point dynamics may turn directly into (wild) chaotic dynamics once a model parameter has crossed a certain bifurcation threshold, implying that even a tiny parameter change may have a dramatic impact on dynamics.

In addition to deterministic agent-based financial market models, stochastic versions also exist. While deterministic models usually generate complex dynamics, thereby mimicking the stylized facts of bubbles and crashes and excess volatility, they usually have difficulties in reproducing the finer details of stock market dynamics. For instance, a prominent feature of actual stock markets is that the distribution of stock market returns possesses fat tails. This is visualized for the Italian and German stock market indices in the third line of Figures 1 and 2 where the distributions of actual returns and normally distributed returns (with identical mean and variance) are compared. Another stylized fact is that stock market returns are virtually unpredictable. Let us look at the penultimate panels of Figures 1 and 2 where the autocorrelation functions of the raw returns are plotted for the first 100 (daily) lags. As we can see, the autocorrelation coefficients are usually insignificant, implying a random walk-like behavior of stock prices. Instead, the bottom panels of Figures 1 and 2 show the autocorrelation functions for absolute returns, revealing significant evidence of volatility clustering and long memory effects. Stochastic agent-based financial market models are able to mimic these statistical

features quite well, see, e.g. Lux and Marchesi (1999), Westerhoff and Dieci (2006), Gaunersdorfer and Hommes (2007) and He and Li (2007). However, these models are typically stochastic versions of smooth agent-based financial market models.

The contribution of this paper is as follows. First, we develop a simple one-dimensional discontinuous piecewise-linear agent-based financial market model. Within our model, prices are driven by the trading activity of heterogeneous speculators who rely on technical and fundamental trading rules. Our model may be regarded as a generalization of models developed in collaboration with Laura Gardini in recent years. Second, we survey some of the analytical results obtained for a number of these (sub-)models. Third, we endeavor to calibrate a stochastic version of our model such that it matches the dynamics of actual financial markets. Despite its piecewise linear nature – or possibly due to this very nature – we believe the model is quite effective in this respect.

As we will see in the sequel, our map consists of three separate linear branches. The dynamics of the model can be investigated analytically as long as the positions of the branches are fixed. In the stochastic version of our model, however, the branches of our map are shifted around erratically such that different dynamics (fixed point dynamics, (quasi-)periodic dynamics, chaotic dynamics or divergent dynamics) are mixed and, as a result, the simulated time series resemble actual time series quite closely. This exercise stresses the importance of establishing analytical results of the deterministic skeleton of a model since they may be the key to understanding the dynamics of more complicated stochastic model versions.

The remainder of our paper is organized as follows. In Section 2, we present our model. In Section 3, we survey the analytical results of special cases of our model. In Section 4, we calibrate a stochastic version of our model and discuss its statistical time series properties. Section 5 concludes our paper and highlights extensions for future work.

2 A simple financial market model

The model we now present may be regarded as a generalization of models developed jointly with Laura Gardini in a series of papers (which will be surveyed in Section 3). In a nutshell, the structure of our model is as follows. Prices adjust with respect to excess demand in the usual way. Excess demand, in turn, is made up of the transactions of four different groups of speculators. First of all, there are so-called type 1 chartists and type 1 fundamentalists. These speculators are always active in the market. Modeling of the chartists was inspired by Day and Huang (1990): chartists believe in the persistence of bull and bear markets and thus buy if prices are high and sell if they are low. Fundamentalists do exactly the opposite. Fundamentalists expect prices to revert towards their fundamentals and thus buy if prices are low and sell if they are high.

Moreover, there are so-called type 2 chartists and type 2 fundamentalists. They behave in the same way as their type 1 counterparts except that they only become active if the price is at least a certain distance away from its fundamental value. It can be argued, for instance, that a certain bubble movement must already have been set in motion to trigger transactions of type 2 chartists (because otherwise they fail to recognize their trading signals). For type 2 fundamentalists it may seem reasonable, due to risk considerations, to only enter the market once there is a real chance and noteworthy potential for mean reversion.

As it turns out, the dynamics of our model is due to a one-dimensional discontinuous piecewise-linear map. In Section 2.1, we present the key building blocks of our model. In Section 2.2, we derive its law of motion.

2.1 The setup

Within our model, prices adjust with respect to excess demand. We use the following (standard) log-linear price adjustment rule, where P is the log of price

$$P_{t+1} = P_t + a(D_t^{C,1} + D_t^{F,1} + D_t^{C,2} + D_t^{F,2}).$$ (1)

The four terms in bracket on the right-hand side of (1) capture the transactions of the four groups of speculators, that is, the transactions of type 1 chartists, type 1 fundamentalists, type 2 chartists and type 2 fundamentalists, respectively. Parameter a is a price adjustment parameter which we set, without loss of generality, equal to $a = 1$. Therefore, (1) states that excess buying drives the price up and excess selling drives it down.

Orders by type 1 chartists are formalized as

$$D_t^{C,1} = \begin{cases} c^{1,a} + c^{1,b}(P_t - F) & for \quad P_t - F \geq 0 \\ -c^{1,c} + c^{1,d}(P_t - F) & for \quad P_t - F < 0 \end{cases}.$$ (2)

The four reaction parameters of (2) are non-negative, i.e. $c^{1,a}, c^{1,b}, c^{1,c}, c^{1,d} \geq 0$. Note first that type 1 chartists optimistically buy (pessimistically sell) if prices are in the bull (bear) market, that is, if log price P is above (below) its log fundamental value F. Reaction parameters $c^{1,a}$ and $c^{1,c}$ capture some general kind of optimism and pessimism, respectively; reaction parameters $c^{1,b}$ and $c^{1,d}$ indicate how aggressively type 1 chartists react to their perceived price signals. Obviously, type 1 chartists may treat bull and bear markets differently with respect to their trading intensity.

Orders by type 1 fundamentalists are written as

$$D_t^{F,1} = \begin{cases} -f^{1,a} + f^{1,b}(F - P_t) & for \quad P_t - F \geq 0 \\ f^{1,c} + f^{1,d}(F - P_t) & for \quad P_t - F < 0 \end{cases},$$ (3)

where the reaction parameters fulfill $f^{1,a}, f^{1,b}, f^{1,c}, f^{1,d} \geq 0$. Type 1 fundamentalists always trade in the opposite direction as type 1 chartists. They sell in an overvalued market and buy in an undervalued market. The trading intensity of type 1 fundamentalists may also differ in bull and bear markets: a certain overvaluation may trigger a larger or smaller absolute order size than an undervaluation of the same size.

Type 2 chartist are only active if prices are at least a certain distance away from their fundamental value. The threshold in the bull market is given by $Z^{C,U}$; the threshold in the bear market is denoted by $Z^{C,D}$. Orders by type 2 chartists may therefore be expressed as

$$D_t^{C,2} = \begin{cases} c^{2,a} + c^{2,b}(P_t - F) & for \quad P_t - F \geq Z^{C,U} \\ 0 & for \quad -Z^{C,D} < P_t - F < Z^{C,U} \\ -c^{2,c} + c^{2,d}(P_t - F) & for \quad P_t - F \leq -Z^{C,D} \end{cases} . \qquad (4)$$

Here we make the following assumptions. We assume that $c^{2,b}, c^{2,d} \geq 0$, i.e. the trading intensity of type 2 chartist increases with the distance between prices and fundamentals. In addition, we assume that $-Z^{C,D} \leq F \leq Z^{C,U}$, i.e. the upper market entry level, indicating a robust bull market, is above the fundamental value and the lower market entry level, indicating a robust bear market, is below the fundamental value. Finally, we assume that $c^{2,a} \geq -c^{2,b}(Z^{C,U} - F)$ and $c^{2,c} \geq -c^{2,d}(Z^{C,D} + F)$. The transactions of type 2 chartists are therefore non-negative in the bull market and non-positive in the bear market. For instance, if $c^{2,a}$ were equal to zero, then transactions of type 2 chartists at the market entry level $Z^{C,U}$ would be given by $c^{2,b}(Z^{C,U} - F) > 0$. Hence, with reaction parameter $c^{2,a}$, transactions of type 2 chartists can, in such a situation, either be increased or decreased, in the latter case down to zero (e.g. $c^{2,a} = -c^{2,b}(Z^{C,U} - F)$).

Orders by type 2 fundamentalists are based on the same principles, i.e. we have

$$D_t^{F,2} = \begin{cases} -f^{2,a} + f^{2,b}(F - P_t) & for \quad P_t - F \geq Z^{F,U} \\ 0 & for \quad -Z^{F,D} < P_t - F < Z^{F,U} \\ f^{2,c} + f^{2,d}(F - P_t) & for \quad P_t - F \leq -Z^{F,D} \end{cases} , \qquad (5)$$

where restrictions $f^{2,b}, f^{2,d} \geq 0$, $f^{2,a} \geq f^{2,b}(F - Z^{F,U})$, $f^{2,c} \geq f^{2,d}(Z^{F,D} + F)$, and $-Z^{F,D} \leq F \leq Z^{F,U}$ apply. In a serious bull market, given by $P - F > Z^{F,U}$, type 2

fundamentalists submit selling orders $-f^{2,a} + f^{2,b}(F-P) < 0$; in a pronounced bear market, given by $P - F < -Z^{F,D}$, they submit buying orders $f^{2,c} + f^{2,d}(P-F) > 0$.

2.2 The model's law of motion

Two simplifying assumptions which we make throughout the rest of the paper are that (i) type 2 chartists and type 2 fundamentalists share the same market entry levels and that (ii) their upper and lower market entry levels are equally distant to the fundamental value. Formally, we thus have $Z = Z^{C,U} = Z^{C,D} = Z^{F,U} = Z^{F,D}$. Moreover, it is convenient to express the model in terms of deviations from the fundamental value by defining $\tilde{P}_t = P_t - F$.

Combining (1) to (5) then yields

$$\tilde{P}_{t+1} = \begin{cases} c^{1,a} - f^{1,a} + c^{2,a} - f^{2,a} + (1 + c^{1,b} - f^{1,b} + c^{2,b} - f^{2,b})\tilde{P}_t & \text{if} \quad \tilde{P}_t \geq z \\ c^{1,a} - f^{1,a} + (1 + c^{1,b} - f^{1,b})\tilde{P}_t & \text{if} \quad 0 \leq \tilde{P}_t < z \\ f^{1,c} - c^{1,c} + (1 + c^{1,d} - f^{1,d})\tilde{P}_t & \text{if} \quad -z < \tilde{P}_t < 0 \\ f^{1,c} - c^{1,c} + f^{2,c} - c^{2,c} + (1 + c^{1,d} - f^{1,d} + c^{2,d} - f^{2,d})\tilde{P}_t & \text{if} \quad \tilde{P}_t \leq -z \end{cases}, \quad (6)$$

which is a one-dimensional discontinuous piecewise-linear map.

To make the notation more convenient, let us introduce

$$\begin{cases} m^1 = c^{1,a} - f^{1,a}, & s^1 = c^{1,b} - f^{1,b}, \\ m^2 = f^{1,c} - c^{1,c}, & s^2 = c^{1,d} - f^{1,d}, \\ m^3 = c^{2,a} - f^{2,a}, & s^3 = c^{2,b} - f^{2,b}, \\ m^4 = f^{2,c} - c^{2,c}, & s^4 = c^{2,d} - f^{2,d}. \end{cases} \quad (7)$$

What can we say about the signs of these eight aggregate parameters? Given the assumptions we have made about the 16 individual reaction parameters, it is clear that each of the eight aggregate parameters can take any value.

With the help of (7), our model can be simplified to

$$\tilde{P}_{t+1} = \begin{cases} m^1 + m^3 + (1 + s^1 + s^3)\tilde{P}_t & if \quad \tilde{P}_t \geq z \\ m^1 + (1 + s^1)\tilde{P}_t & if \quad 0 \leq \tilde{P}_t < z \\ m^2 + (1 + s^2)\tilde{P}_t & if \quad -z < \tilde{P}_t < 0 \\ m^2 + m^4 + (1 + s^2 + s^4)\tilde{P}_t & if \quad \tilde{P}_t \leq -z \end{cases} . \tag{8}$$

The map representing our financial market model is extremely flexible – since there are no restrictions on the eight aggregate parameters, each of its four branches can be positioned everywhere in $(\tilde{P}_{t+1}, \tilde{P}_t)$-space – and thus incorporates a number of potentially interesting subcases. The next section describes some of the analytical results and insights gained so far.

3 Model subcases and analytical results: a brief survey

3.1 Models with two branches

Let us first turn to models which have only two branches. This requires a more fundamental assumption, namely that type 2 speculators are always active, or, expressed in mathematical terms, that $Z = 0$. In addition, let us assume that type 2 speculators only buy and sell fixed amounts of assets, that is $s^3 = s^4 = 0$, and that type 1 speculators do not display any general kind of optimism or pessimism, that is $m^1 = m^2 = 0$. Model (8) then reduces to

$$\tilde{P}_{t+1} = \begin{cases} m^3 + (1 + s^1)\tilde{P}_t & if \quad \tilde{P}_t \geq 0 \\ m^4 + (1 + s^2)\tilde{P}_t & if \quad \tilde{P}_t < 0 \end{cases}, \tag{9}$$

i.e. we obtain a map with two linear branches, having two disjoint intercept parameters, m^3 and m^4, and two slope parameters, s^1 and s^2. Since the aggregate parameters can take any values, many different cases can even be considered for this sub-model. We thus explored model (9) in a series of papers (Tramontana et al. 2010, 2011b, 2011c). On the one hand, we identified scenarios which have the potential to reproduce some stylized facts of financial markets in a certain qualitative sense. On the other hand, we explored these scenarios with a certain mathematical interest since knowledge about piecewise-linear maps is still limited.

In Tramontana et al. (2010), we focused on two scenarios which are able, amongst other things, to generate complex bull and bear market dynamics:

- In the first scenario, type 1 chartists are assumed to be more aggressive than type 1 fundamentalists. This assumption implies that both s^1 and s^2 are strictly positive and that the two linear branches of map (9) thus increase with a slope larger than 1. In addition, the fixed amounts of assets bought or sold by type 2 fundamentalists are assumed to exceed those of type 2 chartists. As a consequence, the intercept of the right branch of the map is negative ($m^3 < 0$) while the intercept of the left branch is positive ($m^4 > 0$). An example of a map with such a parameter constellation is given in the left-hand panel of Figure 3.

- The opposite is true in the second scenario. This means that type 1 fundamentalists are (much more) aggressive than type 1 chartists. At the same time, however, type 2 chartists buy or sell larger amounts of assets than type 2 fundamentalists. For the shape of the map, these assumptions imply that both branches decrease with slopes smaller than -1 ($s^1, s^2 < -2$) and that the right intercept is positive ($m^3 > 0$) while the left intercept is negative ($m^4 < 0$). An example of such a shape of map (9) is shown in the right-hand panel of Figure 3.

One result of this paper is that chaotic dynamics may emerge in both scenarios. Moreover, the chaotic attractor may cover both bull and bear market regions, leading to erratic switches between low and high price levels. Besides boom-and-bust cycles, the model also produces excess volatility.

***** **Figure 3 about here** *****

In the two aforementioned scenarios, there is a certain kind of symmetry. For instance, if type 1 chartists are more aggressive in the bull market than type 1 fundamentalists, then they are also more aggressive in the bear market (the same is true for type 2 speculators). In Tramontana et al. (2010b, c), we relax this assumption. To be precise, we assume that the aggressiveness of type 1 fundamentalists is (slightly) higher than the aggressiveness of type 1 chartists in the bear

market (i.e. $-1 < s^2 < 0$) while it is much higher in the bull market (i.e. $s^1 < -2$). As a result, the slope of the left-hand branch of the map is positive but lower than 1 while the slope of the right-hand branch is negative and lower than -1. Moreover, type 2 fundamentalists dominate type 2 chartists in the bear market, but type 2 chartists dominate type 2 fundamentalists in the bull market. Hence, both intercepts are positive (i.e. $m^3, m^4 > 0$). Again, periodic or chaotic price dynamics with switches between bull and bear markets may be observed. In addition, we study the important role played by the relative position of the intercepts (i.e. $m^3 \overset{>}{\underset{<}{=}} m^4$). Two examples of such maps are given in Figure 4.

***** Figure 4 about here *****

3.2 Models with three branches

Let us now turn to models which have three branches (we now have $Z > 0$). Assume first that $m^1 = m^2 = m^3 = m^4 = 0$, $s^1 = s^2$ and $s^3 = s^4$. The assumptions concerning the intercept parameters imply the absence of any general kind of optimism or pessimism. The assumptions about the slope parameters imply that speculators' aggressiveness is identical in bull and bear markets. We then have the map

$$
\tilde{P}_{t+1} = \begin{cases} (1 + s^1 + s^3)\tilde{P}_t & if & \tilde{P}_t \geq z \\ (1 + s^1)\tilde{P}_t & if & -z < \tilde{P}_t < z \,. \\ (1 + s^1 + s^3)\tilde{P}_t & if & \tilde{P}_t \leq -z \end{cases} \tag{10}
$$

This map, consisting of three branches, was studied in detail in Tramontana et al. (2011a). The shape of the map is depicted in the top left panel of Figure 5. Our main results can be summarized as follows:

- When the slope of the inner branch is, in absolute value, higher than 1 (that is, type 1 chartists dominate over type 1 fundamentalists) and the slopes of the two outer branches are, in

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absolute values, simultaneously lower than 1 (i.e. the joint impact of the two types of fundamentalists dominates the joint impact of the two types of chartists, but not excessively), then bounded trajectories arise in an absorbing interval.

- Surprisingly, only periodic or quasiperiodic motion is possible under this parameter constellation. However, both high periodicity cycles and quasiperiodic motion may, at least for for some parameter combinations, be virtually indistinguishable from chaotic dynamics and may mimic typical bull and bear market patterns.

- When there are cycles, each cycle is structurally unstable. This means that if a cycle of period k exists, then the whole absorbing interval is densely filled with periodic cycles of the same period.

***** **Figure 5 about here** *****

A further interesting scenario concerns situations in which speculators again react symmetrically to bull and bear market price signals ($s^1 = s^2$ and $s^3 = s^4$), but where the intercept parameters are nonzero. In Tramontana et al. (2011d), we consider the case $m^1 = m^2$ and $m^1 = -m^3 = -m^4$. Note that this implies that the general kind of optimism/pessimism of type 1 traders exactly offsets the general kind of optimism/pessimism of type 2 traders. As a result, we obtain the map

$$
\tilde{P}_{t+1} = \begin{cases} (1 + s^1 + s^3)\tilde{P}_t & if \quad \tilde{P}_t \geq z \\ m^1 + (1 + s^1)\tilde{P}_t & if \quad -z < \tilde{P}_t < z, \\ (1 + s^1 + s^3)\tilde{P}_t & if \quad \tilde{P}_t \leq -z \end{cases} \tag{11}
$$

illustrated in the top right panel of Figure 5. Clearly, the difference between map (10) and map (11) is that the inner branch of map (11) has a nonzero intercept. As it turns out, this scenario can also generate endogenous bull and bear market dynamics, both through periodic and chaotic dynamics.

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Finally, another scenario we explored is that in which $m^1 = m^2 = 0$, $m^3 = -m^4$, $s^1 = s^2$

and $s^3 = s^4$. We then obtain the map

$$\tilde{P}_{t+1} = \begin{cases} m^3 + (1 + s^1 + s^3)\tilde{P}_t & if \quad \tilde{P}_t \geq z \\ (1 + s^1)\tilde{P}_t & if \quad -z < \tilde{P}_t < z \\ -m^3 + (1 + s^1 + s^3)\tilde{P}_t & if \quad \tilde{P}_t \leq -z \end{cases} \qquad (12)$$

This map, visualized in the bottom left-hand panel of Figure 5, was studied in Tramontana et al. (2012). One finding is that if this map is buffeted with dynamic noise, it may match the stylized facts of financial markets not only in a qualitative sense, but also in a quantitative sense (yet there are several differences to the dynamics we study in Section 4). Furthermore, it is also worth noting that this map embeds the famous models of Day and Huang (1990) and, in particular, Huang and Day (1993) as special cases. This is seen when the outer two branches are shifted, via parameter m^3, such that they connect with the inner branch (bottom right-hand panel of Figure 5).

4 A stochastic model version

In Section 4 we seek to show that the models in Section 3, which are relatively simple and rely only on a minimum set of economic assumptions, are not only able to replicate certain stylized facts such as bubbles and crashes and excess volatility – they also mimic the finer statistical details of actual stock prices. In Section 4.1, we introduce a stochastic version of our model. Moreover, we discuss the economic meaning of its calibrated parameter setting. In Section 4.2, we show a particular simulation run and explain the functioning of the model. In Section 4.3, we present the results of a large Monte Carlo study to show that our model has the ability to systematically reproduce some important stylized facts of financial markets.

4.1 Specification and calibration of the stochastic model

First of all, let us assume that type 1 speculators treat bull and bear markets symmetrically. Technically, we thus have $m^1 = m^2$ and $s^1 = s^2$, and the map reduces to

$$
\tilde{P}_{t+1} = \begin{cases} m^1 + m^3 + (1 + s^1 + s^3)\tilde{P}_t & if \quad \tilde{P}_t \geq z \\ m^1 + (1 + s^1)\tilde{P}_t & if \quad -z < \tilde{P}_t < z \,. \\ m^1 + m^4 + (1 + s^1 + s^4)\tilde{P}_t & if \quad \tilde{P}_t \leq -z \end{cases}
\tag{13}
$$

Moreover, speculators may randomly deviate from their trading strategies, i.e. all model parameters are from now on regarded as random variables. As a result, both the location and the slope of the model's three branches change randomly over time. Economically, this assumption seems to be quite natural. Speculators do not always follow exactly the same deterministic trading rule. Their mood and aggressiveness depend on a number of factors. Instead of modeling them in detail, we introduce, for simplicity, a degree of randomness to capture unsystematic deviations from the trading rules (2)-(5).

To be precise, we make the following assumptions

$$
\begin{cases} m^1 \sim N(0,\, 0.005), & s^1 \sim N(0.004,\, 0.06), \\ m^3 \sim N(0,\, 0.002), & s^3 \sim N(-0.012,\, 0.03), \\ m^4 \sim N(0,\, 0.002), & s^4 \sim N(-0.024,\, 0.03), \\ z = 0.2, & F = 0. \end{cases}
\tag{14}
$$

Before starting to explaining the economic meaning of (14), we note that (14) is the result of a trial and error calibration process. As we will see in more detail in the sequel, parameter setting (14) is able to generate reasonable dynamics and is thus (indirectly) supported by the data.

What can we say about the economic implications of these assumptions? Let us start with the distributional assumptions concerning m^1, m^2 and m^3 and observe first that all their means are zero, implying that there is no systematic optimism or pessimism amongst speculators. This would be the case, for instance, if we assumed a positive mean for m^1. This could have been interpreted as a general kind of optimism of type 1 chartists, leading to

systematic buying pressure. However, from period to period there may be some unsystematic (random) optimism and pessimism, and the variability of speculators' sentiments is given by the standard deviations of the distributions. Note here that the degree of randomness of type 1 speculators is larger than the degree of randomness of type 2 speculators. This seems to be reasonable since type 2 speculators perceive clearer trading signals than type 1 speculators.

The assumptions about the distribution of the slope parameter of the inner regime imply that type 1 chartists trade, on average, more aggressively than type 1 fundamentalists. However, the joint trading intensity of type 1 and type 2 fundamentalists dominates the joint trading intensity of type 1 and type 2 chartists. Note that the dominance of fundamental trading is highest in the lower regime. By setting the standard deviation in the inner regime higher than in the outer two regimes, we assume that the inner regime is subject to stronger random influences than the two outer regimes. Of course, assuming $z = 0.2$ implies that type 2 speculators enter the market if prices are either 20 percent below or 20 percent above the fundamental value. Without loss of generality, we also set $F = 0$. As a result, we have $\tilde{P}_t = P_t$, thus $\tilde{P}$ can be interpreted as the log price and changes of $\tilde{P}$ as returns. The price is given by $Exp[\tilde{P}]$.[1]

4.2 A typical simulation run

Figure 6, designed in the same way as Figures 1 and 2, shows the outcome of a "typical" simulation run. The first panel of Figure 6 presents the evolution of the price for the first 3392 time steps (this corresponds to the 3392 trading days between 1998 and 2010 in the Italian and German stock markets). Recall that the fundamental value is equal to 1. As can be seen, the model is able to generate bubbles and crashes. For instance, around time step 2000, the market is

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[1] Due to (14), our stochastic model is closely related to the deterministic model of Tramontana et al. (2011a). They exclude random influences and assume that the slopes in the bull and bear market regions are identical. As we will see, analytical results about such deterministic models help explain how our stochastic model works.

overvalued by about 70 percent, and crashes immediately afterwards. The second panel of Figure 6 shows that, despite having a constant fundamental value and there being therefore no fundamental reason for price changes, prices nevertheless fluctuate strongly. Extreme price changes can easily be higher than 5 percent and up to 10 percent; there is also visual evidence of volatility clustering.

The two panels in the third line of Figure 6 compare the distribution of the simulated returns with the results one would obtain from normally distributed returns (based on the same mean and standard deviation). Note first that the distribution of the simulated returns is well behaved: it is unimodal and bell-shaped. Compared to the normal distribution there is, however, less probability mass in the shoulders and more probability mass in the center and the tails. The same feature can be observed in Figures 1 and 2 for the Italian and German stock markets.

The penultimate panel presents the autocorrelation function of the returns. Since the autocorrelation coefficients are insignificant, prices are very close to a random walk. And, indeed, it would be hard to predict future price movements from the top panel. The bottom panel displays the autocorrelation function of absolute returns. These autocorrelation coefficients are highly significant, and decay slowly over time. Even after 100 lags, we find significant autocorrelation coefficients and, thus, evidence of long memory effects. The similarity between the bottom two panels of Figure 6 with the bottom two panels of Figures 1 and 2 is striking.

***** Figure 6 about here *****

Let us next endeavor to understand how the model works. Suppose first that the market is slightly overvalued. As a result, the market is, on average, dominated by type 1 chartists and there is a tendency for prices to rise. However, the dominance of type 1 chartists over type 1 fundamentalists is rather weak and, since the reaction parameters are stochastic, we have (almost) erratic switches between a monotonic convergence towards the fundamental value and a monotonic departure from the fundamental value. Prices are therefore close to being unpredictable. Assume next that prices move away from the fundamental value. If market entry

level z is crossed, type 2 speculators enter the scene. We now have a situation where type 1 and type 2 fundamentalists jointly dominate the trading behavior of type 1 and type 2 chartists. Since the dominance is weak, prices may first move further away from the fundamental value, but are eventually driven back towards more moderate levels. What happens then? Prices could again be pushed upwards; however, due to the stochastic nature of the model, prices may also decrease and even drop below the fundamental value. If this is the case, chartists become pessimistic and tend to drive prices down even further. At some point, it may be the case that the lower market entry level –z is crossed. Then we again have a situation where type 2 speculators become active. Since type 2 fundamentalists are rather aggressive compared to type 2 chartists, prices eventually recover.

Note that the slopes of the branches are close to one in all regimes. On average, the slope of the inner regime is given by 1.004. While there is a tendency for prices to be driven away from fundamentals, this tendency is weak; due to the random nature of the slope, we have a mixture of stable and unstable dynamics. In the upper regime, the average slope is given by 0.992; in the lower regime it is 0.98. The mean reversion pressure is thus higher in the bear market – or, in other words, bear markets are less pronounced and shorter lived than bull markets, which last longer and may become more dramatic. This asymmetry is also visible from Figure 7 where the top panel shows the evolution of log prices in the time domain and the bottom panel shows log prices a time step t+1 versus log prices at time step t. Both panels are based on a time series with 100000 observations (of which every 10th observation has been plotted). Asymmetric, persistent and significant bubble and crash dynamics are clearly visible in both panels. The bottom panel also reveals that the average slope of the three branches is close to one.

***** Figure 7 about here *****

What about the other stylized facts? As prices run away from the fundamental value, both chartists and fundamentalists receive stronger trading signals – their trading rules are just a

linear function of the mispricing. Excess demand in the market therefore also increases, triggering larger price changes. Since bull and bear markets are persistent to some degree, we have regular periods of high volatility, alternating with periods of low volatility where prices are closer to their fundamentals. Periods of high volatility also render the distribution of the returns fat tailed. [2]

4.3 A Monte Carlo study

So far, our analysis has been restricted to one particular simulation run. Now we attempt to evaluate the model in a more serious fashion. To this end, we first estimate certain summary statistics (or moments) of the Italian and German stock market indexes. Then, on the basis of 1000 simulation runs, we check whether our model produces comparable figures for these statistics. Two things should be noted. First, all times series comprise 3392 observations (or 13 years), i.e. the times series are rather short. Second, it seems that the period from 1998 to 2010 was a rather volatile period. Both markets displayed two major crises and thus volatility may have been above its long-run level. For more information about the statistical properties of financial markets and methods to quantify them, see Mantega and Stanley (2000), Cont (2001) and Lux and Ausloos (2002).

With the first two statistics, r_{min} and r_{max}, we look at the most negative and the most positive daily log price change, respectively. As reported in Table 1, both the FTSE MIB and the DAX produced extreme returns of around ±10 percent between 1998 and 2010. In comparison, the extreme returns of our model seem to be somewhat lower on average. For instance, the median negative and positive extreme returns are given b 6 percent and 5 percent, respectively.

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[2] Our model is related to that of Westerhoff and Franke (2011). Technically, their approach consists of only our inner regime. On the one hand, this makes their model simpler than ours. On the other hand, they have to fine-tune their parameters very carefully. To generate bubble dynamics, but to prevent complete price explosions at the same time, they have to set their average slope parameter to slightly below one. Our model is more robust in the sense that it works for a somewhat larger parameter space. For instance, we can allow for a more unstable inner regime when the outer regimes guarantee an eventual end of bubbles.

However, at least 5 percent of the simulation runs produce extreme returns larger than ± 10 percent.

We obtain a similar result when we look at volatility estimates. For this reason, we introduce the volatility estimator $V = (1/T)\sum |P_t - P_{t-1}|$, measuring the average absolute return (T is the sample length, given here by T=3391 observations). The median volatility estimate we obtain for our model is approximately 0.75 percent. For the Italian and German stock markets, we find volatility estimates of around 1 percent. Given this evidence, we may conclude that our model is able to produce excess volatility, and that its volatility is roughly comparable to what we observe in actual stock markets (taking into account that the period 1998 to 2010 was, presumably, more volatile than the long-run average).

To capture the phenomenon of bubbles and crashes, we use the $D = (1/T)\sum |P_t - F_t|$ statistic, which measures the average absolute distance between log prices and log fundamentals. Apparently, this statistic is an indicator of the distortion in the market and quantifies, at least partially, the size of bubbles and crashes. Unfortunately, this statistic cannot be computed for actual markets, at least not as long as there is a reliable indicator of the markets' fundamental values. For our model, however, we find that 90 percent of the simulation runs have a distortion between 8 and 15.5 percent. Hence, bubbles and crashes seem to be present in almost all simulation runs.

***** Table 1 and 2 about here *****

Estimates for the kurtosis K are given by 8.21 for the Italian and 7.11 for the German return distribution. The median value we obtain for our model is 5.38, which is close to these values. Moreover, 95 percent of our simulation runs have a kurtosis of 4.3 or more. Since the kurtosis of a normal distribution is given by 3, this can be regarded as a safe indicator of excess kurtosis. A better indicator of the fat-tailedness of a return distribution is the Hill tail index H, which we compute on the basis of the largest 5 percent of the absolute returns. For actual markets, this statistic tends to hover between 3 and 4 and, indeed, for the Italian and German

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stock markets they are given by 3.18 (FTSE MIB) and 3.06 (DAX). Our model comes quite close to these figures. As we can see, 70 percent of the simulation runs yield Hill estimates between 2.93 and 4.11. In other words, there is ample evidence that our model is able to generate fat-tailed return distributions.

A further important stylized fact of financial markets concerns their unpredictability. Table 1 presents the autocorrelation coefficients of the returns for the first six lags. These autocorrelation coefficients are quite small, and imply that neither the Italian nor the German stock market can be predicted, at least not based on past returns and linear methods. This important feature is matched by our model quite nicely. The autocorrelation coefficients of the simulated returns are essentially insignificant.

Finally, we turn to the markets' tendency to produce volatility clustering and long memory effects, two other closely related universal features of financial markets. The predictability of the volatility can be detected via the autocorrelation coefficients of the absolute returns (which we compute for lags 1, 5, 10, 25, 50 and 100). In real markets, these autocorrelation coefficients are highly significant and decay slowly. Again, this also is the case for the Italian and German markets, and for our artificial market.

It goes without saying that our model is not perfect. However, given the simplicity of our setup, it is surprising to see how closely the data generated by our model comes to actual data. Overall, our model has at least a certain ability to generate bubbles and crashes, excess volatility, fat tails for the distribution of returns, uncorrelated returns and volatility clustering and long memory effects – which are frequently regarded as the most important stylized facts of financial markets.

5 Conclusions

We propose a simple financial market model with heterogeneous interacting speculators. Some speculators believe in the persistence of bull and bear markets and thus optimistically buy if

prices are high and pessimistically sell if prices are low. Other speculators do the contrary, and bet on mean reversion: they buy if markets are undervalued and sell if they are overvalued. While some speculators are always active, other speculators only enter the market if prices are at least a certain distance away from fundamentals. Apart form this (rather natural) assumption, speculators follow piecewise-linear trading rules. Since the dynamics of our model is driven by a discontinuous piecewise-linear map, it is possible to provide a more or less complete analytical study of the model. The main contribution of this paper is to show that a stochastic version of our model – in which we assume that speculators may randomly deviate from their core trading principles – is able to generate quite realistic dynamics. Responsible for this outcome is the fact that the dynamics result from a mixture of different dynamic regimes, including fixed point dynamics, (quasi-)periodic dynamics, chaotic dynamics and divergent dynamics.

Our model may be extended in several directions, three of which are mentioned here. (1) Recall that type 2 chartists and type 2 fundamentalists share the same market entry levels. Relaxing this assumption would lead to a discontinuous piecewise-linear map with five branches, and it would seem worthwhile to invest effort in exploring this more complicated scenario. (2) One assumption of our stochastic model is that all six model parameters change randomly at each time step. A natural question is whether realistic dynamics can also be obtained if there are less frequent parameter changes. For instance, parameters may change only from time to time, either randomly or due to social and/or economic considerations. (3) The shape of our model is extremely flexible. It might therefore be possible to find quite alternative parameter settings which also deliver reasonable dynamics. The question would then be a matter of establishing which parameter setting is the most successful. We hope that Laura Gardini will help us – again – to tackle some of these exciting issues.

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Data	r_{min}	r_{max}	V	D	K	H
FTSE MIB	-0.08	0.11	1.04	-	8.21	3.18
DAX	-0.09	0.11	1.13	-	7.11	3.06

Data	ac_r^1	ac_r^2	ac_r^3	ac_r^4	ac_r^5	ac_r^6
FTSE MIB	0.00	0.00	-0.05	0.09	-0.06	0.01
DAX	-0.02	-0.02	-0.03	0.05	-0.03	-0.03

Data	$ac_{|r|}^1$	$ac_{|r|}^5$	$ac_{|r|}^{10}$	$ac_{|r|}^{25}$	$ac_{|r|}^{50}$	$ac_{|r|}^{100}$
FTSE MIB	0.22	0.29	0.26	0.18	0.13	0.13
DAX	0.20	0.28	0.24	0.18	0.15	0.10

Table 1: Summary statistics for the FTSE MIB and the DAX. Daily data between 1998 and 2010, 3392 observations.

Quantile	r_{min}	r_{max}	V	D	K	H
0.05	-0.10	0.04	0.60	08.0	4.30	2.93
0.25	-0.07	0.05	0.67	10.1	4.79	3.49
0.50	-0.06	0.05	0.74	11.7	5.38	3.81
0.75	-0.05	0.06	0.81	13.2	6.40	4.11
0.95	-0.04	0.10	0.91	15.5	9.76	4.54

Quantile	ac_r^1	ac_r^2	ac_r^3	ac_r^4	ac_r^5	ac_r^6
0.05	-0.05	-0.05	-0.05	-0.05	-0.05	-0.05
0.25	-0.03	-0.03	-0.03	-0.03	-0.02	-0.02
0.50	-0.01	-0.01	-0.01	-0.01	-0.01	-0.01
0.75	0.00	0.00	0.00	0.00	0.00	0.01
0.95	0.03	0.03	0.03	0.03	0.03	0.03

Quantile	$ac_{|r|}^1$	$ac_{|r|}^5$	$ac_{|r|}^{10}$	$ac_{|r|}^{25}$	$ac_{|r|}^{50}$	$ac_{|r|}^{100}$
0.05	0.13	0.11	0.10	0.08	0.05	0.02
0.25	0.17	0.15	0.14	0.12	0.09	0.05
0.50	0.19	0.18	0.17	0.14	0.12	0.07
0.75	0.23	0.22	0.21	0.18	0.14	0.10
0.95	0.30	0.29	0.28	0.24	0.20	0.15

Table 2: Summary statistics for the model. The quantiles of the summary statistics are calculated on the basis of 1000 simulation runs with 3392 observations each.
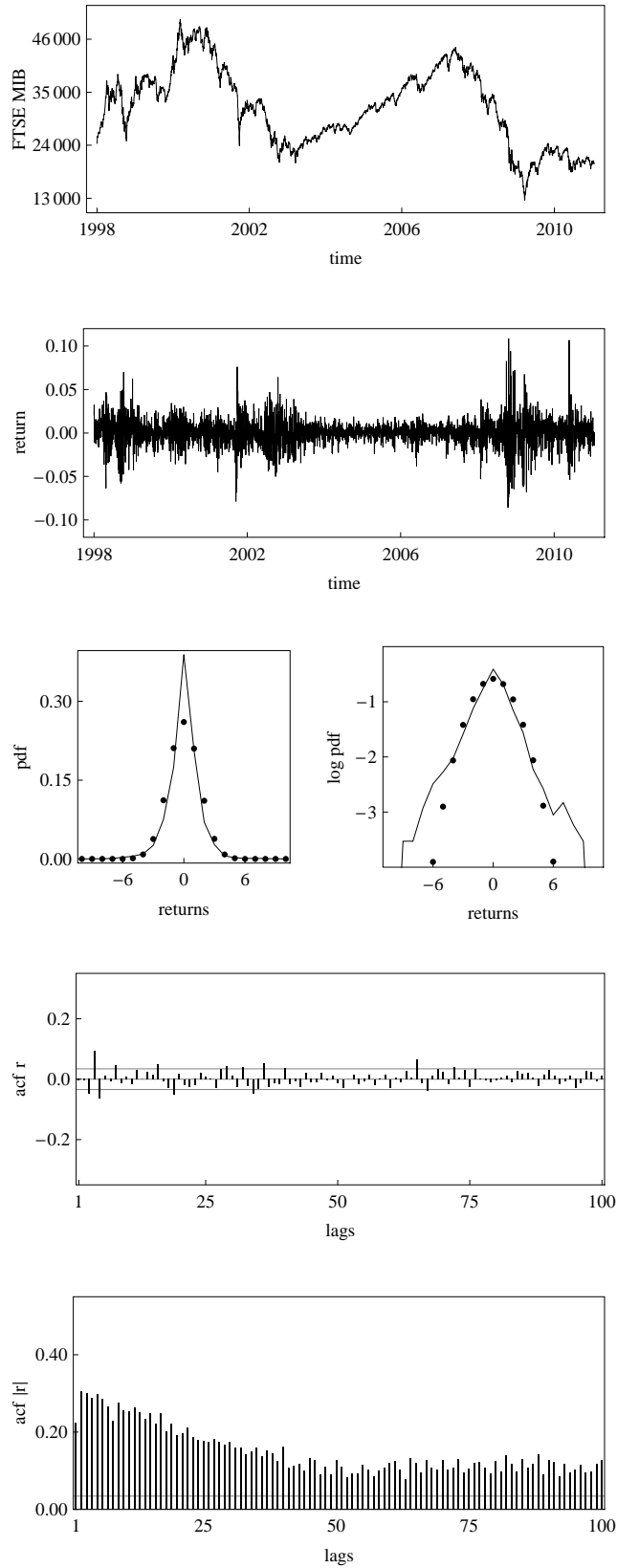
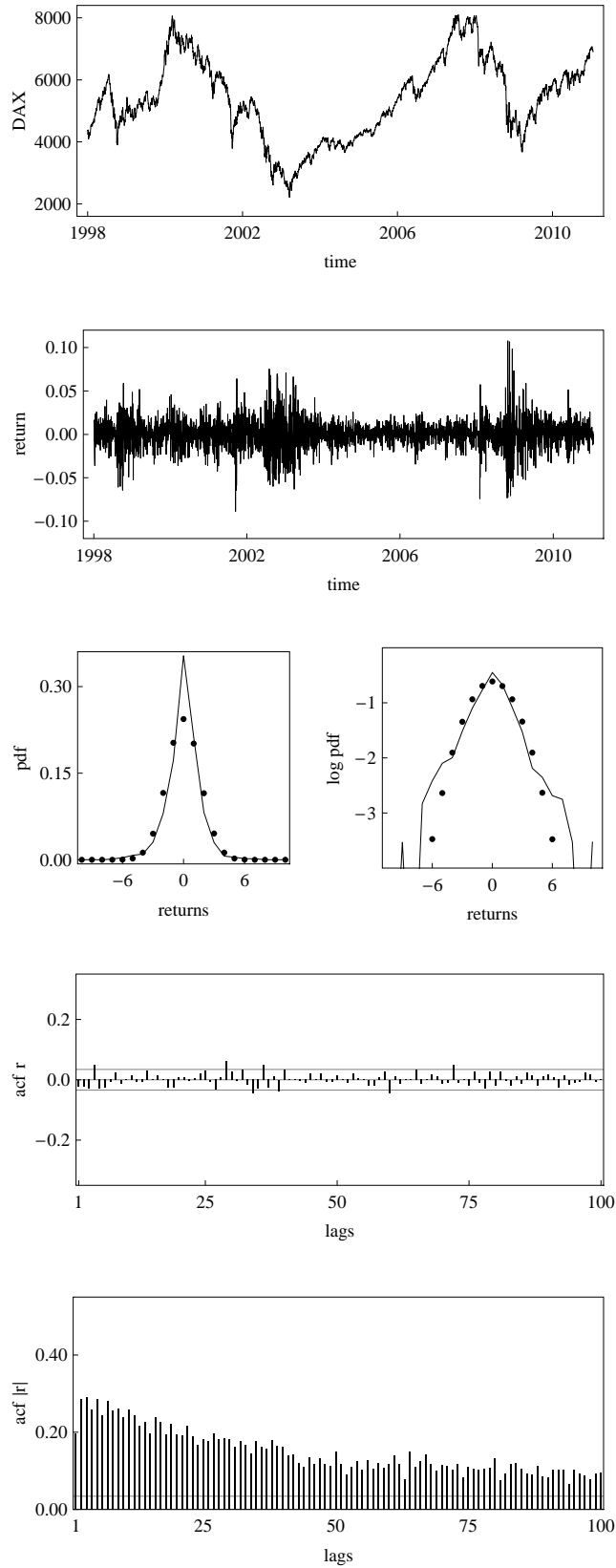Figure 1: The behavior of the FTSE MIB between 1998 and 2010 (3392 daily observations).

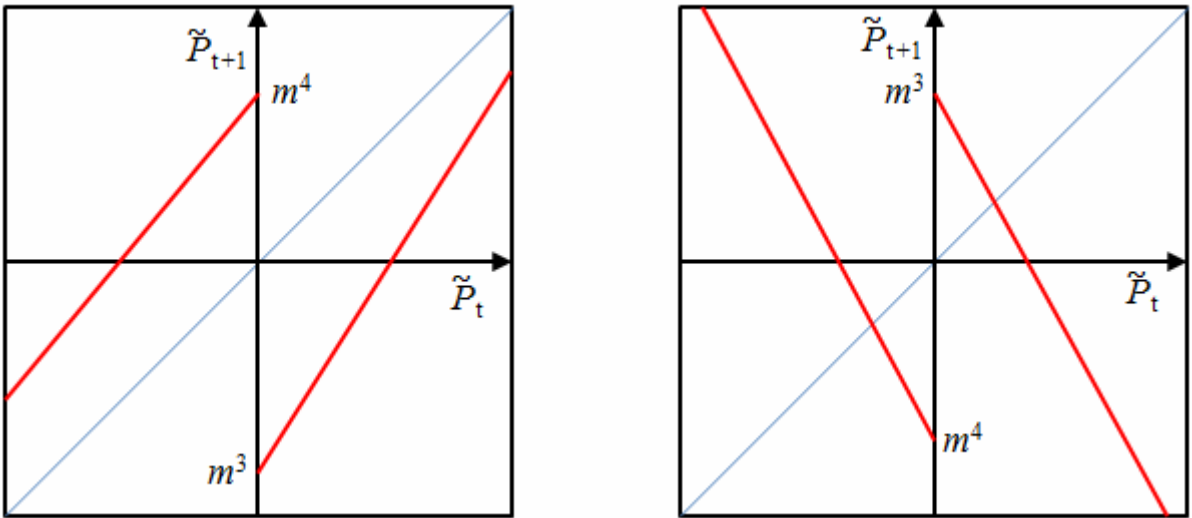Figure 2: The behavior of the DAX between 1998 and 2010 (3392 daily observations).

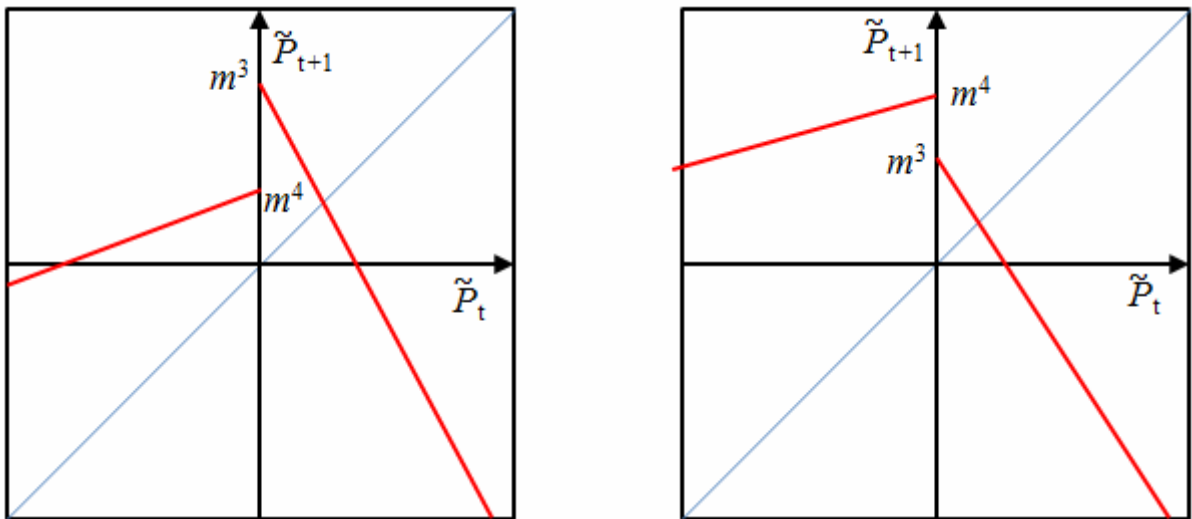Figure 3: Shapes of the map studied in Tramontana et al. (2010).

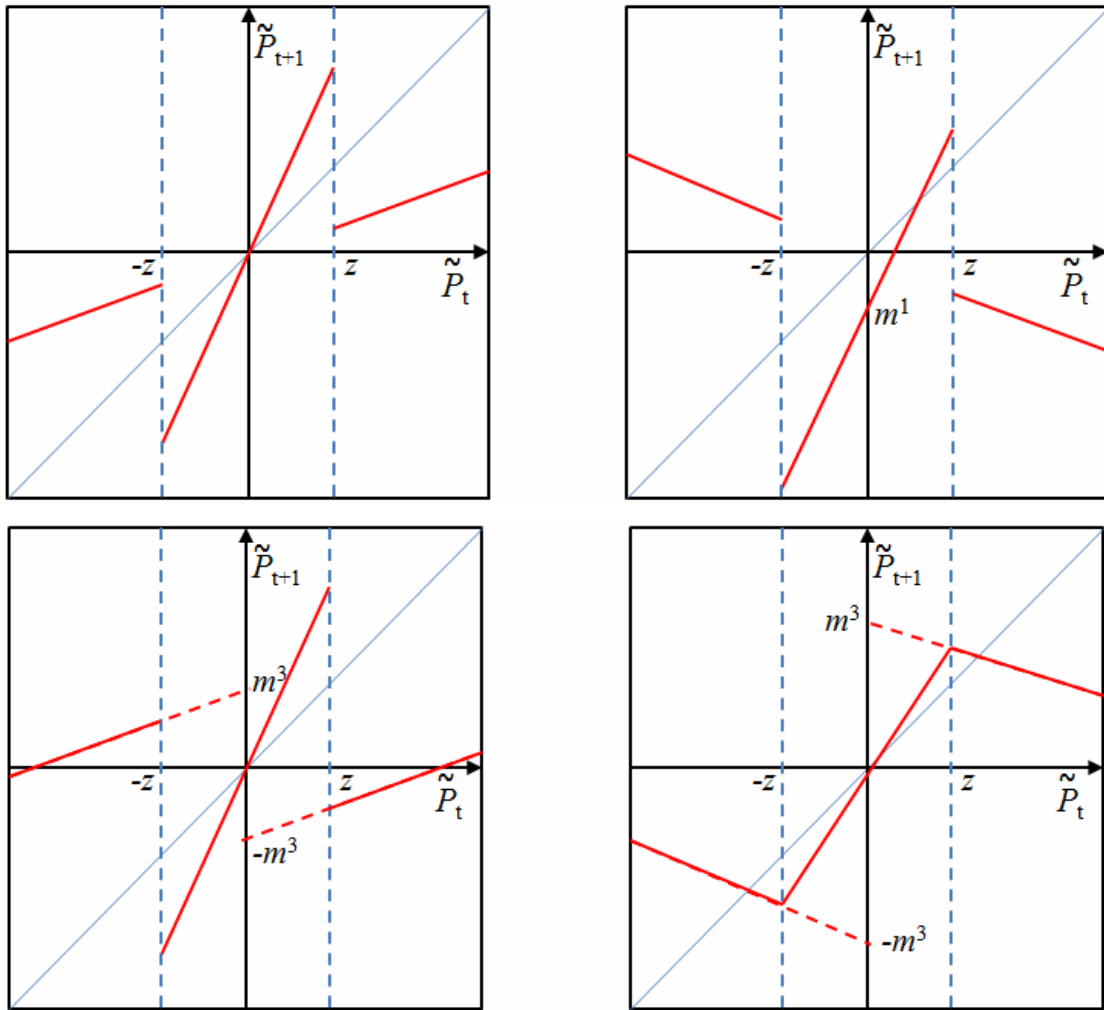Figure 4: Shapes of the map studied in Tramontana et al. (2011b, c).

Figure 5: Shapes of the map studied in Tramontana et al. (2011a, 2011d, 2012).
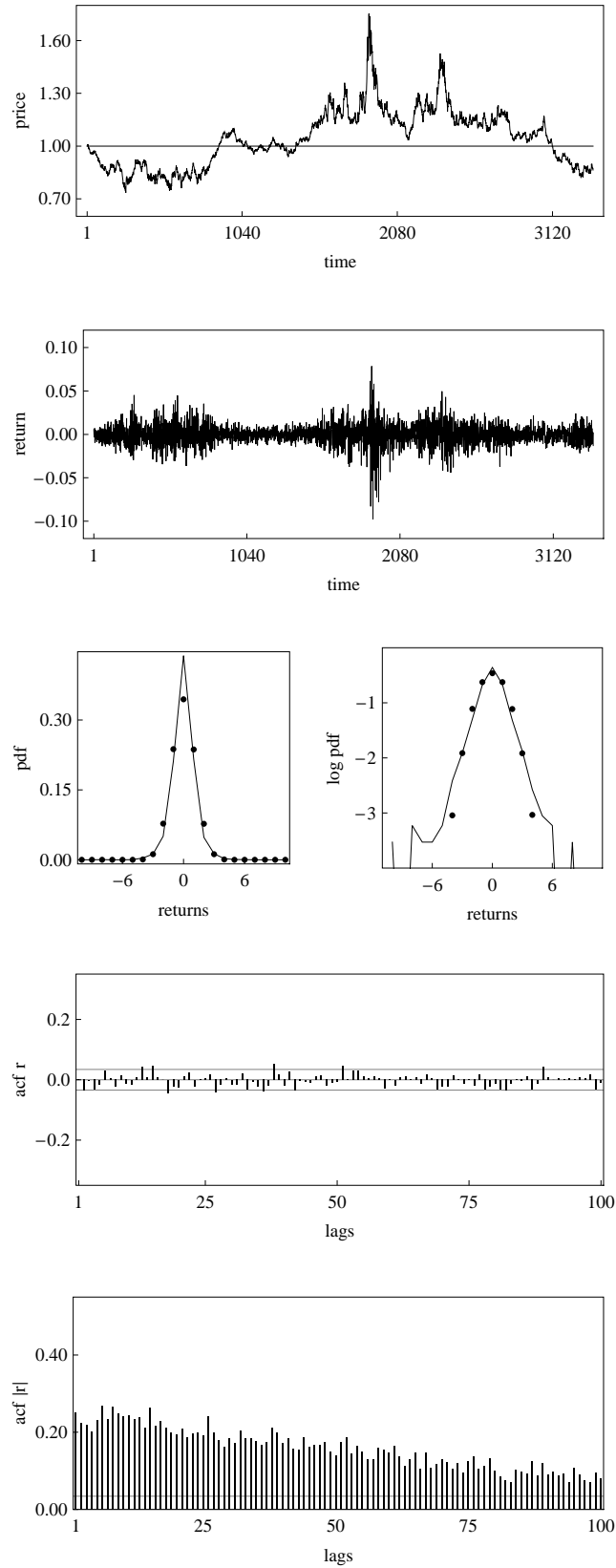
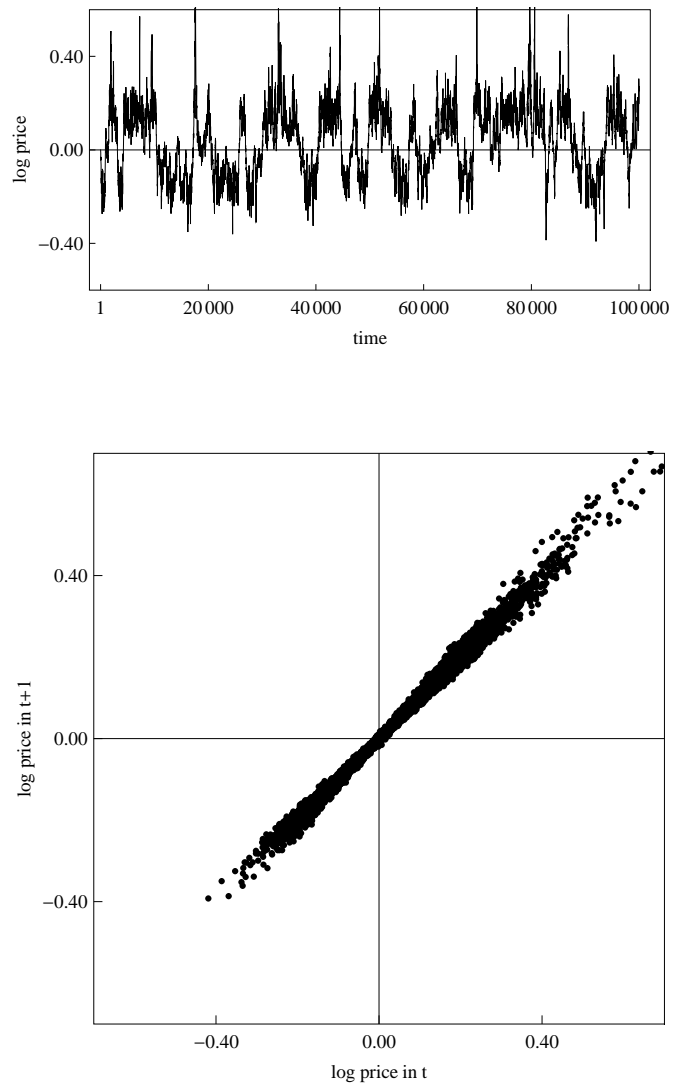Figure 6: A snapshot of the model dynamics (3392 observations).

Figure 7: The model dynamics in the long run (100000 observations).