A simple model of a speculative housing market*

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Abstract

We develop a simple model of a speculative housing market in which the demand for houses is influenced by expectations about future housing prices. Guided by empirical evidence, agents rely on extrapolative and regressive forecasting rules to form their expectations. The relative importance of these competing views evolves over time, subject to market circumstances. As it turns out, the dynamics of our model is driven by a two-dimensional nonlinear map which may display irregular boom and bust housing price cycles, as repeatedly observed in many actual markets. Complex interactions between real and speculative forces play a key role in such dynamic developments.

Keywords

Housing markets; Speculation; Boom and bust cycles; Nonlinear dynamics.

JEL classification

D84; R21; R31.

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1 Introduction

As documented by Shiller (2007, 2008a, 2008b), boom and bust home price cycles have occurred for centuries, yet the recent boom-bust development dwarfs anything seen before. Since the late 1990s, dramatic home price rallies have been observed in countries such as Australia, Canada, China, France, India, Ireland, Italy, Korea, Russia, Spain, the United Kingdom, and the United States. Some of these price movements can be called spectacular. From 1996 to 2008, for instance, real home prices in London nearly tripled. Another impressive example concerns Las Vegas, where real home prices increased by 10 percent in 2003, followed by a 49 percent increase in 2004. For the United States as a whole, real home prices increased by 85 percent between 1997 and 2006. Then the United States' housing market burst and policy makers around the world started to face severe macroeconomic problems.[1]

Shiller (2005, 2008b) argues that this dramatic price increase is hard to explain from an economic point of view since economic fundamentals such as population growth, construction costs, interest rates or real rents do not match up with the observed home price increases. The boom of the early 2000s across cities/countries also suggests that something very broad and general has been at work. This development cannot therefore be linked to factors specific to any of these markets. Shiller (2005, 2008b) concludes that speculative thinking among investors, the use of heuristics such as extrapolative expectations, market psychology in the form of optimism and pessimism, herd behavior and social contagion of new ideas (new era thinking), and positive feedback dynamics are elements that play an important role in determining housing prices. This view contrasts with the standard theoretical approach to house price

[1] Historical data on housing prices is also presented and discussed in Eichholtz (1997), Eitrheim and Erlandsen (2004) and Case (2010).

fluctuations, based on rational expectations and market ‘fundamentals’. In addition, leaving aside Schiller’s recent contribution, the literature on housing price dynamics has long been concerned with the difficulties encountered by the standard approach for explaining, at least qualitatively, the occurrence of booms and busts as well as a number of empirical features of house prices, as briefly summarized in the next subsection.

1.1 A brief review of the related housing literature

Most of the existing studies on housing market dynamics make use of various perfect foresight / rational expectations settings. In this literature, house market fluctuations are essentially viewed as an ‘equilibrium response’ to exogenous factors, namely, as the adjustment process of the price and the stock of housing towards a new steady state position, due to external shocks impinging on economic ‘fundamentals’ of the housing market (such as population, real income, interest rates, …).[2] However, the rational expectations / efficient markets view generally fails to capture the observed dynamics during house price booms and busts and during highly volatile periods, as repeatedly reported by the literature testing for housing market ‘efficiency’ and the presence of bubbles (see., e.g. Clayton 1996, Abraham and Hendershott 1996 and the survey articles by Cho 1996 and Maier and Herath 2009). In particular, a number of empirical studies have presented evidence that changes in house prices are partly predictable, based on past price movements and on other measures capturing the price

[2] Among the several examples, Poterba (1984, 1991) and Mankiw and Weil (1989) focus on the impact of the ‘user cost’ and on demographic changes within an asset-based rational expectations model of the housing market; Ortalo-Magné and Rady (1999) stress the role of income and credit market shocks within a perfect foresight ‘life-cycle’ model; Glaeser and Gyourko (2007) analyze how shocks on demand and construction costs affect house price and quantity adjustments within a no-arbitrage dynamic rational expectations model with endogenous housing supply.

deviation from some fundamental benchmark (such as 'rent-to-price ratios'), which is at odds with market efficiency implied by the rational expectations assumption (see, e.g. Case and Shiller 1989, 1990, Capozza and Seguin 1996, Clayton 1998, and Schindler 2011 for very recent evidence from the S&P/Case-Shiller house price indices). There is also large empirical evidence that house prices tend to display 'momentum' and strong positive serial correlation over short periods, whereas they exhibit mean reversion over longer periods (Capozza et al. 2004, Gao et al. 2009).

Parallel to this, other studies have started to explore the impact of alternative, backward-looking expectations schemes (mostly 'adaptive' or 'naïve' expectations) on house price dynamics (see Wheaton 1999, Malpezzi an Wachter 2005, Glaeser et al. 2008, among others). This branch of literature provides a different perspective on the determinants of house price fluctuations, and shows how the adjustments triggered by exogenous 'fundamental' shocks are largely amplified - and may repeatedly 'overshoot' the new long-run equilibrium - due to the combination of backward-looking expectations on the demand side and the production lags on the supply side. Despite such contributions, there is still a lack of theoretical studies that can suggest simple qualitative explanations for the tendency of housing markets to display dramatic boom / bust episodes.

1.2 Motivation and outline

The goal of our paper is therefore to develop a simple model of a speculative housing market to account for these observations. This may be done, in our view, by explicitly introducing behavioral heterogeneity in speculative demand as well as the possibility of endogenous changes in market sentiment.[3] Our approach is inspired by recent work on agent-based financial market models (see Hommes 2006 and LeBaron 2006 for comprehensive surveys). In these models, the dynamics of financial markets depends on the expectation formation and behavioral rules of boundedly rational heterogeneous interacting agents. As indicated by a number of empirical papers (summarized in Menkhoff and Taylor 2007), financial market participants rely on technical and fundamental trading rules when they determine their orders. Note that extrapolating technical trading rules add a positive feedback to the dynamics of financial markets and thus tend to be destabilizing. By predicting some kind of mean reversion, the effect of fundamental analysis is likely to be stabilizing. Within agent-based financial market models, the impact of these rules is usually time-varying – and it is precisely this that may give rise to complex endogenous dynamics.

For instance, in the models of Kirman (1991, 1993) and Lux (1995, 1997, 1998), agents switch between technical and fundamental analysis due to a herding mechanism, leading to periods where markets are relatively stable (dominance of fundamental analysis) or unstable (dominance of technical analysis). In Brock and Hommes (1997, 1998), the agents select their trading strategies with respect to their past profitability, i.e. this type of model incorporates an evolutionary learning process. Again, endogenous competition between trading strategies may lead to complex price dynamics. Other influential models include Day and Huang (1990), Chiarella (1992), de Grauwe et al.

[3] A number of recent theoretical papers on housing market take heterogeneity into account somehow. For instance, Sommervoll et al. (2010) build a model with buyers, sellers and mortgagees with adaptive expectations, whereas Burnside et al. (2011) develop a model in which agents hold heterogeneous expectations about long-run fundamentals and may change their view because of "social dynamics". Note, however, that the approaches adopted in these models, as well as the underlying concepts of heterogeneity, are very different from ours.

(1993), Chiarella et al. (2002), Westerhoff and Dieci (2006) and de Grauwe and Grimaldi (2006).

Such speculative forces are essential to our model. As pointed out by Shiller (2005, 2008b), the same forces of human psychology that drive international financial markets also have the potential to affect other markets. In particular, this seems to be true for housing markets. Note that by now ample empirical evidence exists to show that human agents generally act in a boundedly rational manner (Kahneman et al. 1986, Smith 1991). Moreover, in many situations people seem to rely on rather simple heuristic principles when asked to forecast economic variables (Hommes et al. 2005, Heemeijer et al. 2009). The model we develop in this paper may thus be regarded as a stylized mathematical representation of what is going on in speculative housing markets. [4] General theoretical and empirical evidence on (nonlinear) speculative bubbles is, for instance, provided by Rosser (1997, 2000).

The structure of our setup is as follows. We assume that housing prices adjust with respect to excess demand in the usual way. The supply of houses is determined by the depreciation of houses and new constructions, which, in turn, depend positively on housing prices. We discriminate between real and speculative demand for houses. As usual, real demand for houses depends negatively on housing prices. Speculative demand for houses is caused by agents' expected future housing prices. For simplicity, agents rely on only two heuristics when they make their predictions. Some agents believe that housing prices will return to a long-run fundamental steady state. However,

[4] Other papers that apply a similar 'heterogeneous interacting agent' approach to the dynamics of housing prices are Leung et al. (2009) and Kouwenberg and Zwinkels (2010). These preliminary studies, however, do not provide analytical results and are mainly concerned with numerical simulation and model calibration.

other agents speculate on the persistence of bull and bear markets. The relative importance of these competing heuristics is due to market circumstances. To be precise, we assume that the more housing prices deviate from the long-run fundamental steady state, the more agents are convinced that some kind of mean reversion is about to set in. The underlying argument is that agents are aware that any bubble will ultimately burst, a situation where mean reversion rules predict the direction of the market movement correctly. A related rule selection scenario is used, for instance, in He and Westerhoff (2005) to understand the cyclical behavior of commodity prices.

The dynamics of our model is due to a two-dimensional nonlinear discrete-time dynamical system. We analytically show that our model may have up to three fixed points. Besides a so-called long-run fundamental steady state, two further steady states may also exist: one located below and one above this value. We determine the region of the parameter space in which the long-run fundamental steady state is locally asymptotically stable. Interestingly, for particular parameter combinations speculative forces can stabilize an otherwise unstable fixed point (via a so-called subcritical flip bifurcation). However, for other possibly more realistic parameter combinations, the impact of speculation is destabilizing. The long-run fundamental steady state of our model may lose its stability via a so-called pitchfork bifurcation, after which two new nonfundamental steady states emerge, or via a so-called Neimark-Sacker bifurcation, after which (quasi-)periodic housing price dynamics set in. The latter scenario becomes more likely when the supply curve is sufficiently sloped (elastic). Finally, we present some numerical examples of boom and bust housing price cycles. These price paths appear to be quite irregular since both real and speculative forces jointly impact on the formation of housing prices and, in turn, realized prices affect agents' demand and supply decisions.

The rest of our paper is organized as follows. In section 2, we introduce a simple housing market model in which speculative forces are absent. In section 3, the model is extended and includes the expectation formation behavior of heterogeneous agents. Section 4 concludes our paper. A number of results are derived in Appendix 1. Appendix 2 provides a brief discussion of alternative formulations of the model.

2 The model without speculation

In this section, we first present our basic housing market model without speculative activity. We also characterize the dynamical system of our model which drives housing prices and the stock of houses, i.e. the model's two state variables.

2.1 Setup

Housing prices evolve with respect to demand and supply. Using a standard linear price adjustment function, the housing price P in period $t+1$ is modeled as

$$P_{t+1} = P_t + a(D_t - S_t)\,, \tag{1}$$

where $a > 0$ is a price adjustment parameter and D and S stand for the total demand and total supply of houses, respectively. Obviously, housing prices increase if demand exceeds supply, and vice versa. Without loss of generality, we set the scaling parameter $a = 1$.

The total demand for houses consists of two components

$$D_t = D_t^R + D_t^S\,, \tag{2}$$

where D_t^R is the real demand for houses and D_t^S is the speculative demand for houses.

The real demand for houses is expressed as

$$D_t^R = b - cP_t \,. \tag{3}$$

Parameters b and c are both positive. As usual, demand depends negatively on the (current) price. In this section, we set $D_t^S = 0$, i.e. we exclude speculative forces for the moment. By basing D_t^R on current housing price P_t only, we model the real demand component as simply as possible and we shift to the second (speculative) component any aspect of the decision process involving expectations about future prices.[5] Note that in theoretical real estate literature, a similar simplifying view has been adopted in a number of studies focusing on the impact of speculative demand on real estate cycles (see, e.g. Malpezzi and Wachter 2005 and references therein).

The supply of houses is given as

$$S_t = S_{t-1} - (1-d)S_{t-1} + eP_t = dS_{t-1} + eP_t \,. \tag{4}$$

The expression $(1-d)$, with $0 < d < 1$, represents the housing depreciation rate. The second term on the right-hand side stands for the construction of new houses. Since $e > 0$, (4) states that the higher the price, the more new houses are built. Note that we exclude any kinds of production lags, for simplicity, nor do we model the price

[5] Put differently, in our model D_t^R represents the desired stock of housing (for given levels of income and population) by people who maximize their utility from 'housing services' and from consumption of alternative goods ('non-housing' consumption), subject to a standard budget constraint. Of course, the possible selling price of houses in the future may, in principle, also be important for these people. This additional aspect would be properly taken into account by modeling agents' utility maximization in a two-period setting (see, e.g. Follain and Dunsky 1997), and price expectations would then play a prominent role by affecting expected utility from second-period wealth. This component is formally shifted to D_t^S in our simplified setup.

expectations of the producers.[6] Assuming that the construction and delivery of new houses for a given period depends on the price construction firms observe in that period implicitly requires that the period is in fact long enough (parameter values chosen in the numerical example in Section 3.3 are compatible with a period of one year). A further possible justification for this assumption comes from the observation that a widespread practice in the housing market consists in pre-selling constructions that are still at the planning or development stages. The housing stock thus includes this ‘pre-sales’ market. This implies that prices prevailing during the development period are at least as important for building decisions as expected future prices (see, e.g. Leung et al. 2007, Chan et al. 2008).

A few final clarifying comments may be pertinent. Note that S and D are ‘stock variables’, namely, the supply of houses S indicates the total stock of houses, whereas D represents total housing demand, in the sense of the desired holding of houses. In the price adjustment equation (1), we match – in each time step – total demand and total supply quantities to determine the next period’s housing price. Alternatively, the model could be formulated in terms of ‘flow variables’ (demand for new houses and new constructions), which would generate qualitatively similar dynamic phenomena. The strict connections between the two formulations become clear within a more general model setup, developed in Appendix 2.

[6] Of course, an interesting extension of our model would be to consider $S_t = dS_{t-1} + eE[P_t]$ and (for instance) $E[P_t] = P_{t-1}$, i.e. new constructions are already planned and executed in period t-1 based on the expected (selling) price for period t, and construction firms hold naïve expectations. Note that such delivery lags represent, in general, further sources of instability (see, e.g. Wheaton 1999). In this particular case, one would end up with a three-dimensional dynamical system which has the same steady states as the present model but an even richer bifurcation structure.

2.2 Dynamical system, fixed point and stability analysis

Recall that $D_t^S = 0$ and $a = 1$. Introducing the auxiliary variable $Z_t = S_{t-1}$, it is possible to reduce (1)-(4) to

$$\begin{cases} P_{t+1} = (1-c-e)P_t - dZ_t + b \\ Z_{t+1} = eP_t + dZ_t \end{cases}, \tag{5}$$

which is a two-dimensional discrete-time linear dynamical system.

By imposing $Z_{t+1} = Z_t = \overline{Z}$ and $P_{t+1} = P_t = \overline{P}$ into (5), we obtain the model's unique fixed point

$$\overline{Z} = \frac{e}{1-d}\overline{P} \tag{6}$$

and

$$\overline{P} = \frac{(1-d)b}{e + c(1-d)}. \tag{7}$$

It follows that $\overline{P}$ and $\overline{Z}$ are always positive. In the following, we call $\overline{P}$ the long-run fundamental steady state of our model, or simply the fundamental value. As revealed by (7), an increase in parameter b leads to an increase in the fundamental value, while an increase in parameters e, c and d yields the opposite, which is, of course, in agreement with common economic sense.

The parameter matrix of our linear map is given as

$$J = \begin{pmatrix} 1-c-e & -d \\ e & d \end{pmatrix}, \tag{8}$$

where $tr = 1-c-e+d$ and $\det = d(1-c)$ denote the trace and the determinant of J, respectively. The fixed point of the linear model (5) is globally asymptotically stable if the following three conditions jointly hold (see, e.g. Medio and Lines 2001 and

Gandolfo 2009): (i) $1+tr+\det>0$, (ii) $1-tr+\det>0$ and (iii) $1-\det>0$. Applying these conditions, we obtain

$$2-\frac{e}{1+d}-c>0, \tag{9}$$

$$c(1-d)+e>0, \tag{10}$$

and

$$1-d+cd>0. \tag{11}$$

Note that the latter two conditions are always true. Inequality (9) implies that the fixed point of our model may lose its stability when parameters c or e increase (demand or supply schedules become more sloped), and when parameter d decreases (the depreciation rate increases). The stability domain of the fixed point is independent of parameter b, the autonomous demand term. Again, this is consistent with economic intuition.

3 The model with speculation

Now we are ready to include speculative activity in our model. Afterwards, in subsection 3.2, we derive the model's dynamical system, its fixed points and the conditions for their local asymptotic stability. Section 3 ends with a few numerical examples of housing price dynamics.

3.1 Speculative demand

We assume that speculative forces entail an extrapolating and a mean reverting component. The relative importance of both components is time-varying since agents change their forecasting rules with respect to market circumstances. For simplicity, we do not track the activities of individual agents in this paper. Our approach may therefore also be interpreted as a model with a boundedly rational representative agent who uses a nonlinear mix of different forecasting rules. The representative agent then updates his/her mix in each time step. Note also that the total demand for houses in our model is simply given as the sum of the real demand for houses and the speculative demand for houses. For instance, if the speculative demand for houses is negative (positive), this decreases (increases) the total demand for houses. A negative speculative demand is simply interpreted as a correction term of the agents' real demand for houses.[7]

Speculative demand driven by the extrapolating component is formalized as

$$D_t^E = f(P_t - \overline{P}) . \qquad (12)$$

The reaction parameter f is positive. When the housing price is above (below) its fundamental value, (12) implies that its followers optimistically (pessimistically) believe in a further price increase (decrease). In other terms, they are confident in the continuation of the housing bubble next period. Accordingly, their speculative demand is positive (negative). This simple yet elegant formulation goes back to Day and Huang (1990), and has been applied in a number of theoretical papers focussing on speculative dynamics (two recent examples are Huang et al. 2010 and Tramontana et al. 2010, while Brock and Hommes 1998 is an earlier example).[8] According to (12), rising prices lead to an increase in demand, i.e. the nature of (12) is indeed extrapolating.

Speculative demand generated by the mean-reverting component is written as

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[7] Our numerical examination, focussing on price and quantity deviations from equilibrium 'fundamental' values, is not affected by parameter *b*, representing the exogenous real demand term. As a consequence, this parameter can always be chosen such that in the original model the total demand for houses is positive in any time step.

[8] Empirical support for this formulation can be found in Boswijk et al. (2007) and Westerhoff and Franke (2011) who estimate small-scale agent-based financial market models which are based on this sort of speculative demand functions.

$$D_t^{MR} = g(\overline{P} - P_t)\,, \tag{13}$$

where g is a positive reaction parameter. For instance, if the housing price is below its fundamental value, agents using (13) expect a price rise and consequently increase their demand for houses.

The total speculative demand is defined as

$$D_t^S = W_t D_t^E + (1 - W_t) D_t^{MR}\,, \tag{14}$$

where W and $1 - W$ stand for the impacts of the extrapolation and mean reversion demand components. Recall that the total demand for houses (2) now consists of real demand for houses (3), buffeted by speculative demand for houses (14).

How do agents choose between the two speculative demand strategies? Following a formulation by He and Westerhoff (2005), which may be traced back to the work of de Grauwe et al. (1993), we assume that the agents update their behavior in every time step with respect to market circumstances.[9] To be precise, the relative impact of extrapolators is formalized as

$$W_t = \frac{1}{1 + h(P_t - \overline{P})^2}\,, \tag{15}$$

where h is a positive parameter. The intuition behind the bell-shaped curve (15) is as follows. Agents seek to exploit price trends (i.e. bull and bear markets). However, the more the price deviates from its fundamental value, the more agents come to the conclusion that a fundamental market correction is about to set in, and they

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[9] Note that the weighting equation proposed by de Grauwe et al. (1993) in an exchange rate setting with heterogeneous agents (and recently adopted also in Bauer et al. 2009) is formally identical to (15), although based on slightly different arguments.

consequently switch to the mean reverting predictor.[10] Note that the higher parameter h, the faster the agents abandon extrapolating behavior as the mispricing increases (i.e. the tails of (15) decline with increasing h). Equation (15) is the most parsimonious way (in terms of dimension of the dynamical system) to model endogenous switching between different behavioral rules. As discussed in Section 1.2, other approaches have often been adopted in the literature, in particular the evolutionary learning approach through 'discrete choice' among alternative rules / predictors, based on past fitness of the rules.[11] However, Westerhoff and Wieland (2010) suggest that there is a close analogy between the discrete choice approach and our formulation, if agents are *forward looking* in their strategy selection and the 'attractiveness' of each rule is properly specified.

Examples for the total demand curve (2), including the speculative component (12)-(15), are drawn in figure 1 for increasing values of the extrapolation parameter f (black lines), together with the real demand (3) (grey lines). A low extrapolation parameter doesn't produce significant nonlinearities and the total demand curve is simply downward-sloped (top left panel). For larger values of f, however, a portion of the demand curve around the fundamental price becomes upward-sloped, i.e. a price rise within this interval results in an increase of the total demand (top right and bottom left

[10] Empirical support for the assumption that fundamentalisms gains in popularity as speculative prices start to deviate from their fundamentals is, for instance, provided by Reitz and Westerhoff (2007), Menkhoff et al. (2009) and Franke and Westerhoff (2011).

[11] In a related paper, Kouwenberg and Zwinkels (2010) use the 'discrete choice' approach of Brock and Hommes (1997, 1998) to model the weights of two speculative demand strategies. According to this approach, agents are boundedly rational in the sense that they tend to select those strategies which have produced a high fitness (measured in terms of realized profits or forecasting errors) in the recent past.

panels). Larger values of f entail an even stronger nonlinearity and increase the amplitude of such an intermediate interval (bottom right panel).

---------- Figure 1 goes about here ----------

3.2 Dynamical system, fixed points and stability analysis

The results we now present are derived in Appendix 1. Let us define the state variables in deviations, $\pi_t = P_t - \overline{P}$ and $\zeta_t = Z_t - \overline{Z}$. It is then possible to rewrite our model as a two-dimensional discrete-time nonlinear dynamical system

$$\begin{cases} \pi_{t+1} = (1-c-e)\pi_t + \dfrac{f\pi_t - gh\pi_t^3}{1+h\pi_t^2} - d\zeta_t \\ \zeta_{t+1} = e\pi_t + d\zeta_t \end{cases}. \quad (16)$$

The dynamical system (16) may have up to three fixed points. For π we find

$$\overline{\pi}_1 = 0 \quad (17)$$

and

$$\overline{\pi}_{2,3} = \pm\sqrt{\frac{(1-d)(f-c)-e}{h(e+(1-d)(c+g))}}. \quad (18)$$

The denominator of (18) is always positive. The latter two fixed points thus only exist if $f > c + e/(1-d) > 0$ (implying a positive nominator). Hence, if the reaction parameter of the extrapolation rule exceeds a certain critical level, the model possesses three fixed points. The housing prices may then permanently be located above or below the fundamental steady state. For the model's second state variable, we obtain $\overline{\zeta}_{1,2,3} = \dfrac{e}{1-d}\overline{\pi}_{1,2,3}$. Accordingly, the equilibrium supply of houses is relatively high (low) if the equilibrium housing price is located in the bull (bear) market. Should the price properly reflect its fundamental value, the supply of houses is as in section 2.

Moreover, the fixed point ($\bar{\pi}_1 = 0$, $\bar{\zeta}_1 = 0$) is locally asymptotically stable if the following inequalities jointly hold

$$f > c + \frac{e}{1+d} - 2, \tag{19}$$

$$f < c + \frac{e}{1-d}, \tag{20}$$

and

$$f < c + \frac{1}{d} - 1. \tag{21}$$

Note first that condition (19) is certainly satisfied within the parameter region in which the model without speculation is stable (this follows immediately from a comparison with condition (9)). However, what is interesting here is that when the first inequality is violated, since f drops below a certain critical level (but the other two inequalities hold), we observe a (subcritical) flip bifurcation. When the second inequality is violated, since f increases (but the other two inequalities hold), we observe a (supercritical) pitchfork bifurcation. Finally, when the third inequality is violated, since f increases (but the other two inequalities hold), we observe a (supercritical) Neimark-Sacker bifurcation.

Let us illustrate these interesting findings. Figure 2 shows four bifurcation diagrams in which we vary the bifurcation parameter f as indicated on the axis. The other parameters are given in table 1.[12] The top panel reveals that the fundamental steady state becomes (locally) attracting if f becomes larger than about 0.019. Hence, speculative forces have a stabilizing impact in this situation.

[12] Note that the rate of depreciation is 2 percent per time period in all of our simulations. Furthermore, assuming that a time period is given with one year, a depreciation rate of 2 percent implies a (reasonable) half-life of a housing unit of roughly 35 years. We thank an anonymous referee for this suggestion.

---------- Table 1 goes about here ----------

However, the picture changes dramatically in the other bifurcation scenarios. The next two panels show the emergence of a supercritical pitchfork bifurcation. If f is equal to 0.615, the fundamental steady state loses its local asymptotic stability and two new nonfundamental steady states appear around it. The two bifurcation diagrams only differ with respect to the chosen initial conditions. Note that housing prices may persistently be higher (second panel) or lower (third panel) than the fundamental steady state. If f increases further, we observe cyclical or even chaotic price dynamics restricted to either the bull or the bear market.[13] For f larger than about 7.27, we find that housing prices endogenously switch between bull and bear market regions (we will discuss this phenomenon in further detail in the next subsection with the help of figure 4, top panels).

---------- Figure 2 goes about here ----------

The bottom panel depicts a supercritical Neimark-Sacker bifurcation. As f exceeds the value of 0.07, the fundamental steady state becomes unstable and instead we observe quasi-periodic motion. Note that the amplitude of the price fluctuations increases with f. The bifurcation diagram also reveals some periodic windows and areas where the dynamics is apparently chaotic (the latter feature will also be revisited in the next subsection, jointly with figure 4, bottom panels).

It is also instructive to represent the region of local asymptotic stability of the fundamental steady state in the plane of the parameters (c, f) by taking the supply

[13] As figure 2 suggests, each of the two nonfundamental steady states changes into a more and more complex attractor, via a sequence of period-doubling bifurcations.

parameter e and the depreciation parameter $(1-d)$ as given.[14] Parameters c and f are particularly important since our analysis stresses the joint effect of real and speculative demand. Note first that each of the three inequalities (19), (20), and (21) results in a half-plane in (c, f) parameter space. The straight lines which bound these half-planes have identical slopes but different intercepts. We can thus easily identify two possible qualitative cases, which we denote as "Case 1" and "Case 2" in figure 3. Since $e > 0$ and $0 < d < 1$, the inequalities $\frac{e}{1-d} > \frac{e}{1+d} - 2$, $\frac{e}{1-d} > 0$ and $\frac{1}{d} - 1 > 0$ always hold. In the qualitative sketches of "Case 1" and "Case 2" it is assumed that parameters e and d are selected in such a way that $e < 2(1+d)$, i.e. $\frac{e}{1+d} - 2 < 0$. The pictures would also remain qualitatively the same in the case $0 \leq \frac{e}{1+d} - 2 < \frac{1}{d} - 1$, except that now the bottom line would lie entirely in the positive quadrant.[15]

However, the qualitative situation $e < 2(1+d)$ is particularly informative. In this case, an interval of positive values of parameter c exists such that (given the selected values of parameters d, e) the steady state of the model without speculation is stable. Such an interval, given as $(0, 2 - e/(1+d))$, is represented in bold on the horizontal axis. In the opposite case, the model without speculative demand would be unstable for any value of c.

---------- Figure 3 goes about here ----------

[14] Note that these parameters capture the supply-side of the economy.

[15] Note that for $\frac{e}{1+d} - 2 \geq \frac{1}{d} - 1$, or, equivalently, $e \geq (1+d)^2 / d$, the steady state is unstable for any combination (c, f). We do not consider this case in figure 3.

Let us now compare "Case 1" with "Case 2". The bifurcation scenario sketched in "Case 1" occurs when the following condition (which is easily interpreted graphically) holds

$$\frac{e}{1-d} > \frac{1}{d} - 1, \tag{22}$$

or, equivalently, $e > (1-d)^2 / d$. "Case 1" thus occurs if the supply is sufficiently elastic or if the depreciation rate is small enough. By contrast, the condition for "Case 2" is $e < (1-d)^2 / d$ for which the supply must be more inelastic or the depreciation rate higher.

In "Case 1" we observe a Neimark-Sacker bifurcation if the extrapolating component of the demand (governed by parameter f) is sufficiently strong. However, as discussed before, it is also possible for a (subcritical) flip bifurcation to occur if parameter f becomes small enough (assuming that parameter c is outside the range of stability of the model without speculative demand). The latter bifurcation can be regarded as the possibility that a sufficiently strong component of extrapolative demand stabilizes an otherwise unstable steady state via a reverse subcritical flip bifurcation.

The above considerations about the Flip bifurcation also remain true in "Case 2". However, instead of a Neimark-Sacker bifurcation there is a pitchfork bifurcation when the extrapolative demand becomes strong enough. The latter gives rise to two further locally stable nonfundamental steady states. The destabilizing impact of speculative demand may therefore give rise to different 'local' bifurcations scenarios, associated to different characteristics of the supply-side of the economy. The above analysis also highlights the stabilizing role played by (elasticity of) real demand, at least for what concerns local stability. The stability plots in figure 3 and conditions (19)-(21) show that a higher value of parameter c (within the range $0 \le c \le 2 - e/(1+d)$) can enlarge the stability domain of parameter f, thus partially offsetting the effect of speculative demand.

3.3 Some numerical examples

The goal of this subsection is to study the types of dynamic behavior our model may produce in greater detail. In particular, we will investigate two examples, both given in figure 4. The first example, presented in the first two panels, corresponds to the pitchfork bifurcation scenario depicted in the second and third panel of figure 2, for which we now assume $f = 7.28$. The first (second) panel in figure 4 shows housing prices (the stock of houses) in deviations from the fundamental steady state. As can be seen, our model is able to generate complex bull and bear market dynamics. Housing prices fluctuate in an intricate manner for some time above their long-run steady state value. Then, out of the blue, housing markets crash, after which housing prices fluctuate below their steady state value. Overall, the duration of bull and bear market episodes is quite unpredictable (this becomes more evident if one inspects longer simulation runs). Moreover, the stock of houses adjusts gradually over time: it increases during bull markets and decreases during bear markets. If a bull (bear) market lasts long enough, the stock of houses eventually exceeds (drops below) its long-run equilibrium value. For shorter bull (bear) market episodes this is not necessarily the case.

A second example of intricate housing price cycles is given in the third and fourth panels of figure 4. The underlying parameter setting is that used in the bottom panel of figure 2 with $f = 6$, i.e. after the Neimark-Sacker bifurcation. Now the dynamics is characterized by irregular bubbles and crashes. Housing prices may increase for a number of periods. At some point, however, a correction sets in, which usually leads to a severe crash. It is interesting to note that the model is able to generate boom and bust cycles with quite different appearances. Both the duration and amplitude of the cycles vary to some degree.[16] This is also mirrored in the development of the stock of houses.

---------- Figure 4 goes about here ----------

Recall that real home prices in London more than doubled from 1983 to 1988 and then fell 47 percent by 1996. From 1996 to 2008, real home prices in London nearly tripled again. However, the latter development was briefly interrupted between mid-2004 to mid-2005, when real home prices decreased by about 6 percent. This downturn was then quickly reversed with annual growth rates of 9 percent. According to Shiller (2007), such irregularities in boom and bust cycles are hard to explain with standard economic thinking since one would expect a price dip to mark the end of a bubble and lead directly to a crash. We find it worthwhile to point out that our model may endogenously generate such price dynamics.

In the panels of figure 5, we present from top to bottom π_t versus ζ_t, π_t versus ζ_{t-1}, and π_t versus π_{t-1}, respectively. The left-hand (right-hand) panels are based on the dynamics of the top (bottom) panels of figure 4. The appearance of strange attractors underlines the complexity of the dynamics our model is able to produce. However, these panels also indicate a number of striking differences between the

[16] The historical housing price data provided by Shiller (http://www.econ.yale.edu/~shiller/) share, in a qualitative sense, some phenomena with our simulated housing price data. In particular, the period from 1890 to 1975 seems to be characterized by "bull and bear market dynamics" whereas the period from 1975 to 2010 displays more pronounced "boom and bust cycles". However, the disaggregated data presented in Case (2010) is more ragged than Shiller's nationwide data.

dynamics of the ‘pitchfork’ scenario (top panels of figure 4) and the ‘Neimark-Sacker’ scenario (bottom panels of figure 4). In all three panels on the left-hand side, we can make out a positive relation between the plotted variables, that is, we observe that prices tend to increase with the current and previous period’s stock of houses and that prices display some kind of persistence (i.e. high prices tend to be followed by high prices, and likewise for low prices). Note that positive serial correlation between housing prices has been widely reported by the empirical literature, as discussed Section 1.1. With respect to the persistence of prices, we find a similar effect on the right-hand side. [17] However, the relation between the price of houses and the stock of houses is now negative.

---------- Figure 5 goes about here ----------

Let us finally try to sketch the events that may drive housing price bubbles. Suppose, for instance, that prices are slightly above the fundamental value. Then the majority of agents is optimistic and expects a price increase. As a result, demand for houses increases and prices are pushed upwards for a certain period. During this process, however, the market appears to be increasingly overvalued and agents start to switch to mean reversion expectations. Then some kind of adjustment towards the fundamental value sets in. If this adjustment is rather strong, we may even observe a crash. Otherwise, the rally continues after the price dip. Such possibly different scenarios depend on the interactions between real demand, speculative forces, and the existing stock of houses. Roughly speaking, as long as housing prices are high, new constructions increase the stock of houses. During a downwards movement, however,

[17] Note, however, that in the bottom panels of figure 5 the relation between house prices in subsequent periods is reversed from positive to negative when considering price ranges that deviate from the benchmark fundamental considerably.

the demand for houses may be considerably lower than the supply of houses, amplifying any price reduction.

This story is in line with the conclusion of Shiller (2008b), who argues that there has been a tendency in many cities for home prices to rise and crash, but to show little long-term trend. Prices rise while people are optimistic, but forces are set in motion for them to crash when they get too high. In our model, these forces contain a speculative component (dominance of regressive expectations) as well as a real component (excess supply of houses).

3.4 The impact of supply-side behavior

So far, our setup focuses only on speculative demand behavior. On the supply side, our assumptions imply that the amount of new constructions, being proportional to the current market price, is strictly positive at each time step[18], irrespective of the fact that the price level may be too low, and of the possible accumulation of unsold houses from previous periods. One consequence of this working assumption is that price deviations from the fundamental value display some sort of symmetry, i.e. positive and negative deviations are of roughly the same average size, as is clear from the numerical examples of the previous subsection. We now briefly[19] provide an example in which the linear supply function (4) is changed into the following piecewise linear supply function

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[18] Note that in the steady state solution of the model, the amount of new constructions exactly offsets the depreciation of the existing stock of houses. More generally, the stock of houses may decline also in the presence of new constructions, if the latter are not sufficient to compensate for depreciation.

[19] We leave an analytical study as well as a systematic numerical investigation of this model for future work. However, as noted by one of the anonymous referees, it may be interesting to study this model in more detail, in particular how the steady states and their stability domain depend on the model parameters.

$$S_t = \begin{cases} dS_{t-1} & \text{if } P_t < P_{\min} \text{ and } P_t - P_{t-1} < -\gamma \\ dS_{t-1} + eP_t & \text{otherwise} \end{cases}, \quad (23)$$

with $\gamma \geq 0$. Through (23), construction of new houses stops whenever prices are ‘too low’ and a certain amount of excess supply (i.e. unsold houses) has accumulated from the previous period (note from (1) that condition $P_t - P_{t-1} < -\gamma$ is equivalent to $S_{t-1} - D_{t-1} > \gamma / a$). Hence, (23) accounts for the tendency to postpone construction of new houses if the market circumstances appear not promising. This is a simple but realistic behavioral assumption.

Despite the fact that the overall law of motion of the system now becomes discontinuous, this additional ‘speculative’ component on the supply side does not seem to alter the key qualitative features of the model, i.e. its ability to generate complex housing price dynamics. Rather, the model now tends to produce more asymmetric price fluctuations. An example for this is presented in figure 6 where we use the Neimark-Sacker parameter setting and set, in addition, $\gamma = 0$ and $P_{\min} = \overline{P} - 1$. As can bee seen from the top panel, the price path is now characterized by an increased frequency of high realization (compared with the situation displayed in the third panel of figure 4). This is even more obvious from a comparison of the price sample distributions, changing from almost symmetric in the original setting (bottom left panel of figure 6, corresponding to the third panel of figure 4) to right skewed in the new setting (bottom right panel of figure 6). Moreover, average and median prices, which are close to the fundamental value (i.e. close to zero) in the original setting increase for the new supply function. Overall, these results suggest that the introduction of the option to delay the building of new housing in some market circumstances interacts with speculative and real activity on the demand side and makes the price dynamics even more jagged.

---------- Figure 6 goes about here ----------

4 Conclusions

In this paper we develop a simple model of a speculative housing market to improve our understanding of boom and bust housing prices cycles. The key feature of our model is that the demand for houses is affected by speculative forces. While some agents are convinced that housing prices converge towards their long-run fundamental value, other agents optimistically (pessimistically) believe in the persistence of bull and bear market dynamics. Since agents change their prediction strategies from time to time with respect to market circumstances, our model is nonlinear. We find that such speculative forces, interacting with the real demand and the evolving stock of houses, may imply the (co)existence of (strange) attractors, and can lead to complex price dynamics. In particular, our model has the potential to generate intricate bubbles and crashes, as observed recently in many housing price markets around the world.

We see our model as a first formal behavioral contribution to understand the intricate dynamics of housing markets. Clearly, more work is needed to better understand the functioning of housing markets and to be able to derive effective stabilization programs for housing markets. Broadly speaking, the analysis of our stylized model has highlighted the impact of the exogenous parameters characterizing the *real* demand and supply-side of the economy, not only on the long-run 'fundamental equilibrium', but also on the likely types of bifurcation occurring when speculative forces tend to prevail, as well as on the nature of price fluctuations. In this respect, it is clear that such parameters could be affected by policy interventions in various ways, thus providing policy makers the opportunity to dampen the effects of speculative activity. Finally, we would like to point out a number of directions into which our model could be developed. First, one could assume that the speculative demand component is explicitly driven by past housing price changes. Second, one may consider that speculators base their choice of forecasting rules on criteria such as past realized profits or prediction errors, thus strengthening the connections between our approach and the ‘evolutionary finance’ literature. Third, there are various ways how one could make the supply side of the model more realistic. For instance, one could introduce production lags and price expectations on the part of the producers. Alternatively, one could model the speculative activity of the supply side and the developers’ decisions in more detail. Fourth, one could also try to embed a behavioral, speculative housing market approach such as ours into a macroeconomic model. Finally, in the future it will become increasingly important to bring behavioral models to the data, and one could thus try to calibrate or estimate models such as ours. Given the importance of housing markets and their role for the real economy we hope that more effort is directed towards these important issues.

References

Abraham, J. and Hendershott, P. (1996): Bubbles in metropolitan housing markets. Journal of Housing Research, 7, 191–207.

Bauer, C., DeGrauwe, P. Reitz, S. (2009): Exchange rate dynamics in a target zone - A heterogeneous expectations approach. Journal of Economic Dynamics and Control, 33, 329-344.

Boswijk, P., Hommes, C. and Manzan, S. (2007): Behavioral heterogeneity in stock prices. Journal of Economic Dynamics and Control, 31, 1938-1970.

Brock, W. and Hommes, C. (1997): A rational route to randomness. Econometrica, 65, 1059-1095.

Brock, W. and Hommes, C. (1998): Heterogeneous beliefs and routes to chaos in a simple asset pricing model. Journal of Economic Dynamics Control, 22, 1235-1274.

Burnside, C., Eichenbaum, M. and Rebelo, S. (2011): Understanding booms and busts in housing markets, NBER Working Paper 16734.

Capozza, D. and Seguin, P. (1996): Expectations, efficiency, and euphoria in the housing market. Regional Science and Urban Economics, 26, 369-386.

Capozza, D., Hendershott, P. and Mack, C. (2004): An anatomy of price dynamics in illiquid markets: analysis and evidence from local housing markets. Real Estate Economics, 32, 1–32.

Case, K. (2010): Housing, land and the economic Crisis. Land Lines, 22, 8-13.

Case, K. and Shiller, R. (1989): The efficiency of the market for single-family homes. American Economic Review, 79, 125 – 137.

Case, K. and Shiller, R. (1990): Forecasting prices and excess returns in the housing market. Journal of the American Real Estate and Urban Economics Association, 18, 253–273.

Chan, S.H., Fang, F., Yang, J. (2008): Presales, financing constraints, and developers' production decisions. Journal of Real Estate Research, 30, 345-375.

Chiarella, C. (1992): The dynamics of speculative behavior. Annals of Operations Research, 37, 101-123.

Chiarella, C, Dieci, R, and Gardini, L. (2002): Speculative behaviour and complex asset price dynamics: A global analysis. Journal of Economic Behavior and Organization, 49, 173-197.

Cho, M. (1996): House price dynamics: a survey of theoretical and empirical issues, Journal of Housing Research, 7, 145-172.

Clayton, J. (1996): Rational expectations, market fundamentals and housing price volatility. Real Estate Economics, 24, 441-470.

Clayton, J. (1998): Further evidence on real estate market efficiency. Journal of Real Estate Research, 15, 41–57.

Day, R. and Huang, W. (1990): Bulls, bears and market sheep. Journal of Economic Behavior and Organization, 14, 299-329.

De Grauwe, P., Dewachter, H. and Embrechts, M. (1993): Exchange rate theory – chaotic models of foreign exchange markets. Blackwell, Oxford.

De Grauwe, P. and M. Grimaldi (2006): Exchange rate puzzles: a tale of switching attractors. European Economic Review, 50, 1-33.

Eichholtz, P. (1997): A long run house price index: the Herengracht index, 1628-1973. Real Estate Economics, 25, 175-192.

Eitrheim, Ø. and Erlandsen, S. (2004): House price indices for Norway 1819-2003. In Eitrheim, Ø., Klovland, J. T. and Qvigstad, J. F. (eds.): Historical Monetary Statistics for Norway 1819-2003. Norges Bank Occasional Paper no. 35, 349-375. Norges Bank, Oslo.

Follain, J.R. and Dunsky, R.M. (1997): The demand for mortgage debt and the income tax, Journal of Housing Research, 8, 155–199.

Franke, R. and Westerhoff, F. (2011): Estimation of a structural stochastic volatility model of asset pricing. Computational Economics, in press

Gandolfo, G. (2009): Economic dynamics, 4th edition. Springer-Verlag, Berlin.

Gao, A., Lin, Z. and Na, C.F. (2009): Housing market dynamics: evidence of mean reversion and downward rigidity. Journal of Housing Economics, 18, 256-266.

Glaeser, E. and Gyourko, J. (2007): Housing dynamics. HIER Discussion Paper 2137.

Glaeser, E., Gyourko, J. and Saiz, A. (2008): Housing supply and housing bubbles. Journal of Urban Economics, 64, 198-217.

He, X.-Z. and Westerhoff, F. (2005): Commodity markets, price limiters and speculative price dynamics. Journal of Economic Dynamics and Control, 29, 1577-1596.

Heemeijer, P. Hommes, C. Sonnemans, J. and Tuinstra, J. (2009): Price stability and volatility in markets with positive and negative expectations feedback: an experimental investigation. Journal of Economic Dynamics and Control, 33, 1052-1072.

Hommes, C. (2006): Heterogeneous agent models in economics and finance. In: Tesfatsion, L. and Judd, K. (eds.): Handbook of computational economics Vol. 2: Agent-based computational economics. North-Holland, Amsterdam, 1107-1186.

Hommes, C., Sonnemans, J., Tuinstra, J. and van de Velden, H. (2005): Coordination of expectations in asset pricing experiments. Review of Financial Studies, 18, 955-980.

Huang,W., Zheng, H., Chia, W.-M. (2010): Financial crises and interacting heterogeneous agents. Journal of Economic Dynamics and Control, 34, 1105-1122.

Kahneman, D., P. Slovic, A. Tversky (1986): Judgment under uncertainty: heuristics and biases. Cambridge.

Kirman, A. (1991): Epidemics of opinion and speculative bubbles in financial markets. In: Taylor, M. (Ed.): Money and financial markets. Blackwell, Oxford, 354-368.

Kirman, A. (1993): Ants, rationality, and recruitment. Quarterly Journal of Economics, 108, 137-156.

Kouwenberg, R. and Zwinkels, R. (2010): Chasing trends in the US housing market. Working Paper, Erasmus University Rotterdam.

LeBaron, B. (2006): Agent-based computational finance. In: Tesfatsion, L. and Judd, K. (Eds.): Handbook of computational economics Vol. 2: Agent-based computational economics. North-Holland, Amsterdam, 1187-1233.

Leung, B., Hui, E., Seabrooke, B. (2007): Pricing of presale properties with asymmetric information problems. Journal of Real Estate Portfolio Management, 13, 139-152.

Leung, A., Xu, J. and Tsui, W. (2009): A heterogeneous boundedly rational expectation model for housing market. Applied Mathematics and Mechanics (Engl. Ed.), 30, 1305-1316.

Lux, T. (1995): Herd behavior, bubbles and crashes. Economic Journal, 105, 881-896.

Lux, T. (1997): Time variation of second moments from a noise trader/infection model. Journal of Economic Dynamics and Control, 22, 1-38.

Lux, T. (1998): The socio-economic dynamics of speculative markets: Interacting agents, chaos, and the fat tails of return distributions. Journal of Economic Behavior and Organization, 33, 143-165.

Maier, G. and Herath, S. (2009): Real estate market efficiency—a survey of literature. SRE-Discussion Paper 2009–07, WU Wien.

Malpezzi, S. and Wachter, S. (2005): The role of speculation in real estate cycles. Journal of Real Estate Literature, 13, 143-164.

Mankiw, G. and Weil, D. (1989): The baby boom, the baby bust, and the housing market. Regional Science and Urban Economics, 19, 235-258.

Medio, A. and Lines, M. (2001): Nonlinear dynamics: A primer. Cambridge University Press, Cambridge.

Menkhoff, L. and Taylor, M. (2007): The obstinate passion of foreign exchange professionals: technical analysis. Journal of Economic Literature, 45, 936-972.

Menkhoff, L., Rebitzky, R.R. and Schröder, M. (2009): Heterogeneity in exchange rate expectations: Evidence on the chartist-fundamentalist approach. Journal of Economic Behavior and Organization, 70, 241-252.

Ortalo-Magné, F. and Rady, S. (1999): Boom in, bust out: young households and the housing price cycle. European Economic Review, 43, 755-766.

Poterba, J. (1984): Tax subsidies to owner-occupied housing: an asset market approach. Quarterly Journal of Economics, 99, 729-752.

Poterba, J. (1991): House price dynamics: the role of tax policy and demography. Brookings Papers on Economic Activity, 2, 143-203.

Reitz, S. and Westerhoff, F. (2007): Commodity price cycles and heterogeneous speculators: A STAR-GARCH model. Empirical Economics, 33, 231-244.

Rosser, J. B., Jr. (1997): Speculations on nonlinear speculative bubbles. Nonlinear Dynamics, Psychology and Life Sciences, 1, 275-300.

Rosser, J. B., Jr. (2000): From catastrophe to chaos: a general theory of economic discontinuities. Kluwer Academic Publishers: Boston.

Schindler, F. (2011): Predictability and persistence of the price movements of the S&P/Case-Shiller house price indices. Journal of Real Estate Finance and Economics, DOI: 10.1007/s11146-011-9316-1.

Shiller, R. (2005): Irrational exuberance (second edition). Princeton University Press, Princeton.

Shiller, R. (2007): Understanding recent trends in house prices and home ownership. Cowles Foundation Discussion Paper No. 1630. Yale University, New Haven.

Shiller, R. (2008a): Historical turning points in real estate. Eastern Economic Journal, 34, 1-13.

Shiller, R. (2008b): The subprime solution. Princeton University Press, Princeton.

Smith, V. (1991): Papers in experimental economics. Cambridge University Press, Cambridge.

Sommervoll, D., Borgersen, T. and Wennemo, T. (2010): Endogenous housing market cycles, Journal of Banking and Finance, 34, 557-567.

Tramontana, F., Westerhoff, F. and Gardini, L. (2010): On the complicated price dynamics of a simple one-dimensional discontinuous financial market model with heterogeneous interacting traders. Journal of Economic Behavior and Organization, 74, 187-205.

Westerhoff, F. and Dieci, R. (2006): The effectiveness of Keynes-Tobin transaction taxes when heterogeneous agents can trade in different markets: A behavioral finance approach. Journal of Economic Dynamics and Control, 30, 293-322.

Westerhoff, F. and Wieland, C. (2010): A behavioral cobweb model with heterogeneous speculators. Economic Modelling, 27, 1136-1143.

Westerhoff, F. and Franke, R. (2011): Converse trading strategies, intrinsic noise and the stylized facts of financial markets. Quantitative Finance, in press.

Wheaton, W. (1999): Real estate “cycles”: some fundamentals”. Real Estate Economics, 27, 209-230.

Appendix 1

In this appendix, we derive the two-dimensional nonlinear dynamical system of the full model, its fixed points, the parameter region for which the model's fundamental steady state is locally asymptotically stable, and necessary conditions for the emergence of a flip, a pitchfork, and a Neimark-Sacker bifurcation, respectively. A theoretical treatment of linear and nonlinear dynamical systems is provided by Gandolfo (2009) and Medio and Lines (2001), among others.

Note first that, by setting $\pi_t = P_t - \overline{P}$ and $\zeta_t = Z_t - \overline{Z}$, the two-dimensional linear dynamical system (5) for the model without speculation may be rewritten in terms of deviations from the fundamental steady state as

$$\begin{cases} \pi_{t+1} = (1-c-e)\pi_t - d\zeta_t \\ \zeta_{t+1} = e\pi_t + d\zeta_t \end{cases} . \tag{A1}$$

By now including the speculative demand term, we easily obtain the following two-dimensional nonlinear dynamical system in (π_t, ζ_t)

$$\begin{cases} \pi_{t+1} = (1-c-e)\pi_t + \dfrac{f\pi_t - gh\pi_t^3}{1+h\pi_t^2} - d\zeta_t \\ \zeta_{t+1} = e\pi_t + d\zeta_t \end{cases} . \tag{A2}$$

Inserting $(\pi_{t+1}, \zeta_{t+1}) = (\pi_t, \zeta_t) = (\overline{\pi}, \overline{\zeta})$ into (A2), the three fixed points

$$(\overline{\pi}_1, \overline{\zeta}_1) = (0,0) \tag{A3}$$

and

$$(\overline{\pi}_{2,3}, \overline{\zeta}_{2,3}) = \left(\pm\sqrt{\frac{(1-d)(f-c)-e}{h(e+(1-d)(c+g))}} \,,\quad \frac{e}{1-d}\overline{\pi}_{2,3} \right) \tag{A4}$$

can be calculated. Since the denominator of $\overline{\pi}_{2,3}$ is always positive, the fixed points $(\overline{\pi}_{2,3}, \overline{\zeta}_{2,3})$ only exist if $(1-d)(f-c)-e > 0$.

The Jacobian matrix of our model, evaluated at the steady state $(\bar{\pi}_1, \bar{\zeta}_1) = (0,0)$, reads

$$J = \begin{pmatrix} 1-c-e+f & -d \\ e & d \end{pmatrix}, \tag{A5}$$

where $tr = 1-c-e+d+f$ and $\det = d(1-c+f)$ stand for the trace and determinant of J, respectively. A set of necessary and sufficient conditions for both eigenvalues of J to be smaller than one in modulus (which implies a locally asymptotically stable steady state) is given by (i) $1+tr+\det > 0$, (ii) $1-tr+\det > 0$ and (iii) $1-\det > 0$, respectively. After some simple transformations, this yields

$$f > c + \frac{e}{1+d} - 2, \tag{A6}$$

$$f < c + \frac{e}{1-d}, \tag{A7}$$

and

$$f < c + \frac{1}{d} - 1. \tag{A8}$$

Observe that for $f = 0$, (A6) to (A8) are identical to (9) to (11). In this case, (A7) and (A8) would always be fulfilled. For $f > 0$, however, (A6) is less restrictive than (9), while (A7) and (A8) impose stronger restrictions. Note also that (A6)-(A8) are independent of parameters b and h.

Violation of the first, second and third inequality (the remaining two inequalities hold) represents a necessary condition for the emergence of a flip, pitchfork and Neimark-Sacker bifurcation, respectively. In connection with supporting numerical evidence, this is usually regarded as strong evidence. Figure 2 furthermore reveals that the flip bifurcation is of the subcritical case whereas the pitchfork and Neimark-Sacker bifurcations are of the supercritical type.

Appendix 2

In this appendix, we outline a more general model that includes as particular cases both the simplified formulation in 'stock' variables (adopted in this paper) and a formulation in pure 'flow' variables (new home demand and new constructions). Here we denote by x_t the demand for houses and by y_t the supply of houses in period *t*. The price adjusts to the excess demand in the usual manner, i.e.

$$P_t = P_{t-1} + a(x_{t-1} - y_{t-1}), \quad a = 1. \tag{A9}$$

Demand and supply x_t and y_t (that are now regarded as 'flow' variables) include, in general, part of *unsatisfied demand* (x_{t-1}^B) and *unsold houses* (y_{t-1}^U) from the previous period, respectively. We neglect the speculative demand term for the moment. Demand in period *t* is specified as

$$x_t = \hat{b} - cP_t + \alpha x_{t-1}^B. \tag{A10}$$

Demand x_t thus consists of new demand $\hat{b} - cP_t$ and *backlogged* demand, here simply modeled as a fraction α, $0 \le \alpha \le 1$, of the demand that has remained unsatisfied in the previous period. Supply (i.e. houses for sale) in period *t* is defined as:

$$y_t = eP_t + \beta d y_{t-1}^U, \tag{A11}$$

including new constructions, eP_t, and a fraction β, $0 \le \beta \le 1$, of unsold houses from the previous period (note that the term $\beta d y_{t-1}^U$ takes depreciation into account). By definition, in each period *t* we have:

$$x_t^B = \max(x_t - y_t, 0), \quad \text{i.e.} \quad x_t = x_t^B + \min(x_t, y_t), \tag{A12}$$

$$y_t^U = \max(y_t - x_t, 0), \quad \text{i.e.} \quad y_t = y_t^U + \min(x_t, y_t), \tag{A13}$$

where the term $\min(x_t, y_t)$ represents the amount of houses sold (or, equivalently, of demand satisfied) in period *t*. Equations (A9)-(A11), together with identities (A12) and (A13) form a dynamical system expressed in flow variables, which takes backlogged demand and unsold houses into account. This model can be transformed into an equivalent model where 'stock' variables (the existing stock of houses and the desired *holding* of houses), rather than flow variables, are matched in each period. Note first that the quantity:

$$Q_t := \sum_{k=0}^{t} \min(x_k, y_k) d^{t-k}, \tag{A14}$$

or recursively

$$Q_t = \min(x_t, y_t) + dQ_{t-1} = x_t - x_t^B + dQ_{t-1} = y_t - y_t^U + dQ_{t-1}, \tag{A15}$$

represents the cumulated amount of houses sold in the current and previous rounds, by taking depreciation into account. By defining demand and supply in terms of stock (denoted by D_t and S_t, respectively) as follows:

$$D_t := x_t + dQ_{t-1} = x_t^B + Q_t, \tag{A16}$$

$$S_t := y_t + dQ_{t-1} = y_t^U + Q_t, \tag{A17}$$

dynamical system (A9)-(A13) can be rewritten as a three-dimensional system in the state variables P_t, D_t and S_t:

$$P_t = P_{t-1} + a(D_{t-1} - S_{t-1}), \quad a = 1, \tag{A18}$$

$$D_t = \hat{b} - cP_t + \alpha D_{t-1} - (\alpha - d)Q_{t-1}, \tag{A19}$$

$$S_t = eP_t + \beta dS_{t-1} + (1-\beta)dQ_{t-1}, \tag{A20}$$

where $Q_t = \min(D_t, S_t)$, which turns out to be non-differentiable.[20] Easy computations demonstrate that dynamical system (A18)-(A20) admits a unique steady state, the coordinates of which are specified as follows:

$$\overline{P} = \frac{\hat{b}}{c+e} \ , \tag{A21}$$

$$\overline{D} = \overline{S}(=\overline{Q}) = \frac{e\overline{P}}{1-d} = \frac{\hat{b}e}{(c+e)(1-d)} \ . \tag{A22}$$

In order to check that the stationary levels (A22) of supply and demand, as well as the steady state price (A21), correspond in fact to those obtained in equations (6) and (7), it is enough to change the coordinates of the autonomous demand term, by defining the new parameter b (the one we adopt in the paper) as follows,

$$b := \hat{b} + d\overline{S} = \hat{b} + \frac{\hat{b}de}{(c+e)(1-d)} = \hat{b}\frac{c(1-d)+e}{(c+e)(1-d)}, \tag{A23}$$

as can be shown by simple computations.

Next, the model with speculation can be obtained by adding a demand term D_t^S, identical to (14), to the right-hand side of equation (A19). As numerical simulations suggest, also this more general model produces a transition to complex boom and bust cycles, once extrapolative demand becomes strong enough.

Finally, the following significant particular cases give rise to two simplified models. First, the case $\alpha = \beta = 0$ (unfilled demand and unsold houses are not translated to the next period) can be reduced to the following one-dimensional model in 'pure' flows:

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[20] We have obtained (A18)-(A20) starting from (A9)-(A11) and using the fact that, for any t: $x_t - y_t = D_t - S_t$, $x_t^B = D_t - Q_t$, $y_t^U = S_t - Q_t$ (which follow immediately from (A16)-(A17)).

$$P_t = P_{t-1} + a(D_{t-1} - S_{t-1}) = P_{t-1} + a(\hat{b} - (c+e)P_{t-1} + D_{t-1}^S), \quad a = 1, \tag{A24}$$

where the speculative demand D_{t-1}^S is itself a cubic-type function of P_{t-1} (via (12)-(15)). As can be shown, this model generates a pitchfork scenario for the steady states, followed by a sequence of bifurcations leading to chaotic dynamics, very similar to that illustrated in the paper.

Second, the case $\alpha = \beta = 1$ (unsold houses and unsatisfied demand are entirely shifted to the next period) leads to a three-dimensional model formed by a price adjustment equation identical to (A18) and by the two equations

$$D_t = (b_{t-1} - cP_t) + D_t^S, \tag{A25}$$

$$S_t = dS_{t-1} + eP_t. \tag{A26}$$

In equation (A25) the real demand $b_{t-1} - cP_t$, regarded as a function of P_t, has an 'autonomous' component that depends on the state of the system at time $t-1$, namely, $b_{t-1} := \hat{b} + D_{t-1} - (1-d)\min(D_{t-1}, S_{t-1})$. In order to reduce the dimension of the system and to preserve differentiability, in the paper we replace the time varying term b_{t-1} in equation (A25) with the constant parameter b defined by (A23). The latter is nothing else than the steady-state value of b_{t-1}, i.e. $b := \hat{b} + \bar{S} - (1-d)\bar{S} = \hat{b} + d\bar{S}$. Such a simplification results in the two-dimensional model studied in the paper.

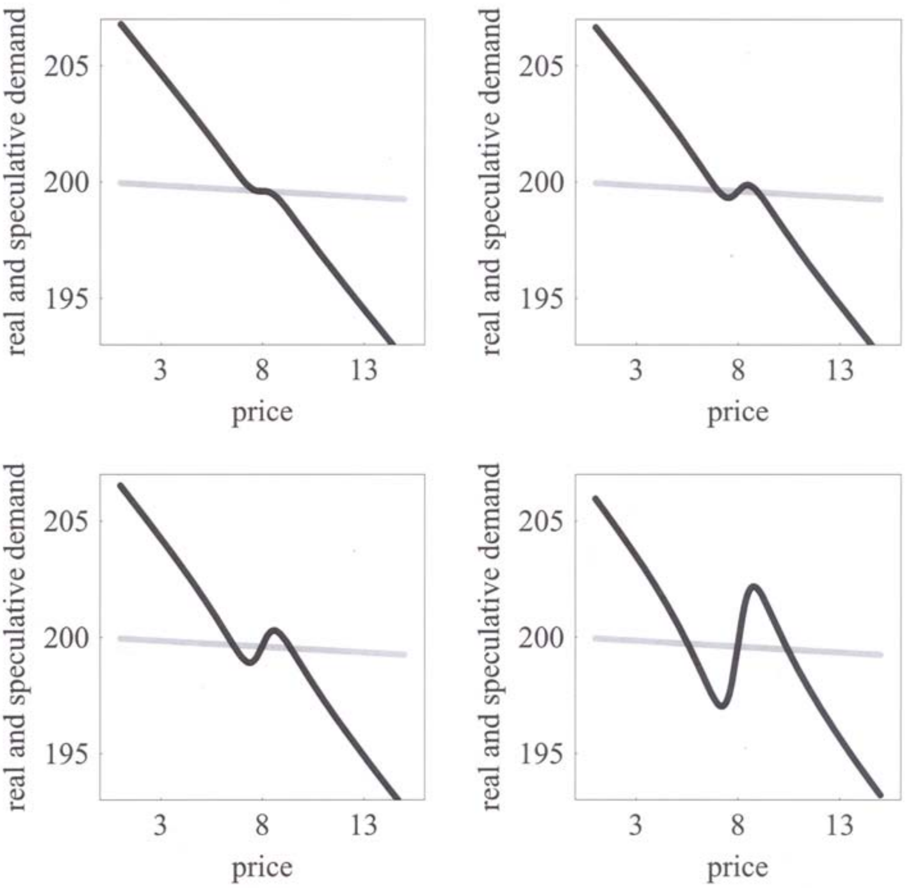

Figure 1: Real demand (grey lines) and total demand (black lines). Parameters *c*, *d*, *e*, *g* and *h* as in the 'Neimark-Sacker' scenario in table 1 and $b = 200$. The extrapolation parameter *f* is equal to 0.05 (top left), 1 (top right), 2 (bottom left) and 6 (bottom right), respectively.

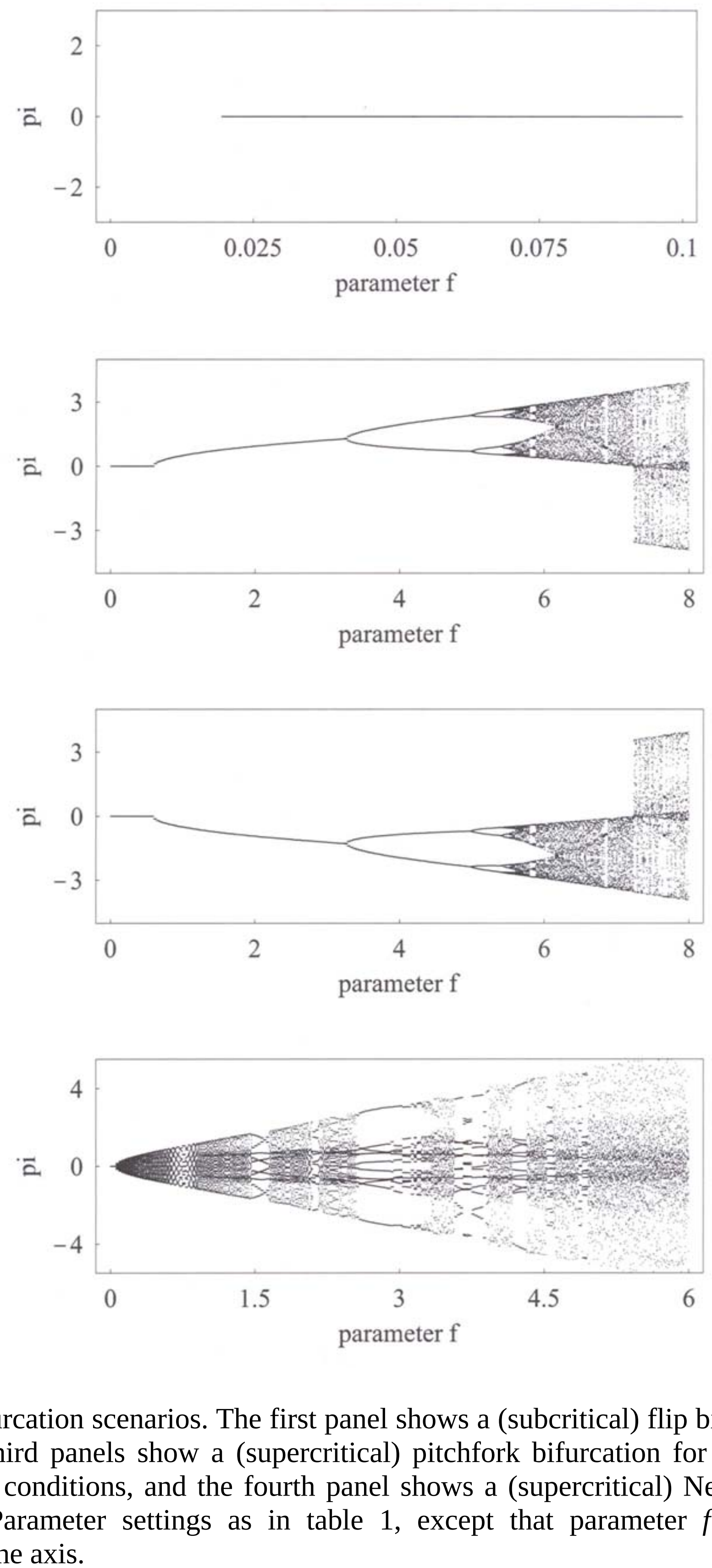

Figure 2: Bifurcation scenarios. The first panel shows a (subcritical) flip bifurcation, the second and third panels show a (supercritical) pitchfork bifurcation for two different sets of initial conditions, and the fourth panel shows a (supercritical) Neimark-Sacker bifurcation. Parameter settings as in table 1, except that parameter *f* is varied as indicated on the axis.

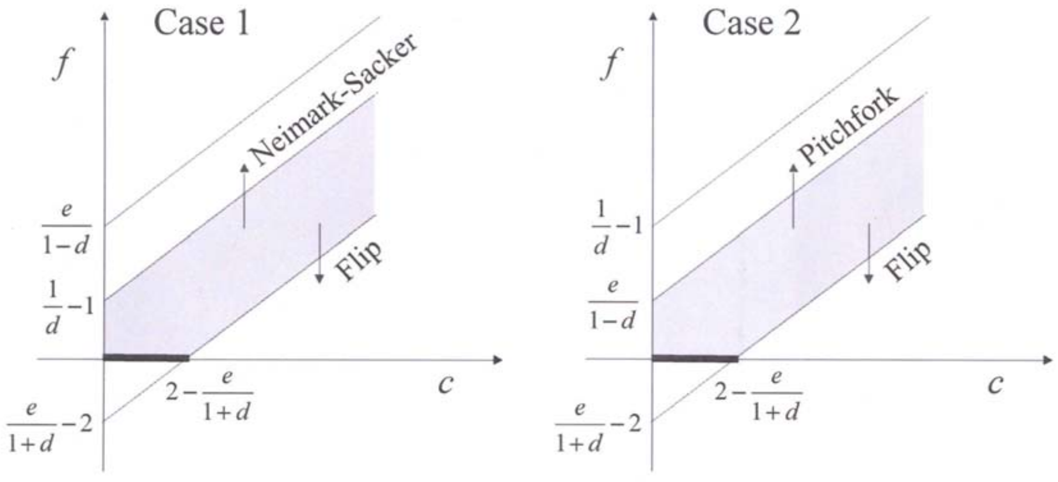

Figure 3: Qualitative representation of the local asymptotic stability region of the 'fundamental steady state' in the plane of the parameters (c, f), taking parameters e and d as given. The left (right) panel depicts "Case 1" ("Case 2"), i.e. a situation where the supply curve is relatively sloped (flat).

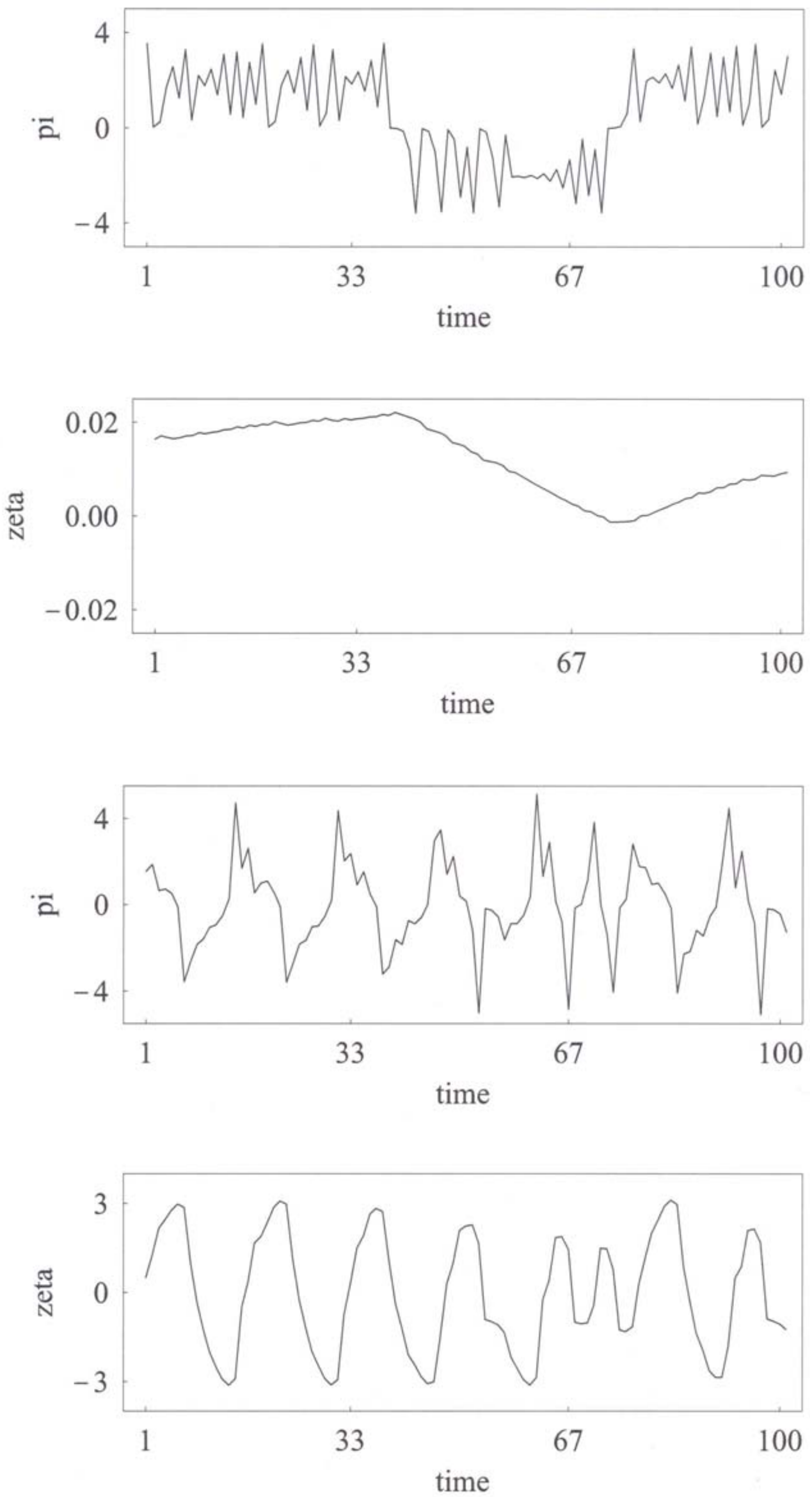

Figure 4: Some snapshots of the model dynamics. The top two panels show examples of persistent bull and bear market dynamics (housing prices and the stock of houses, both in deviations from the fundamental steady state). The parameter setting corresponds to the pitchfork bifurcation scenario, as indicated in table 1. The bottom two panels show examples of bubbles and crashes (housing prices and the stock of houses, in deviations from the fundamental steady state). The parameter setting corresponds to the Neimark-Sacker bifurcation scenario, as indicated in table 1.

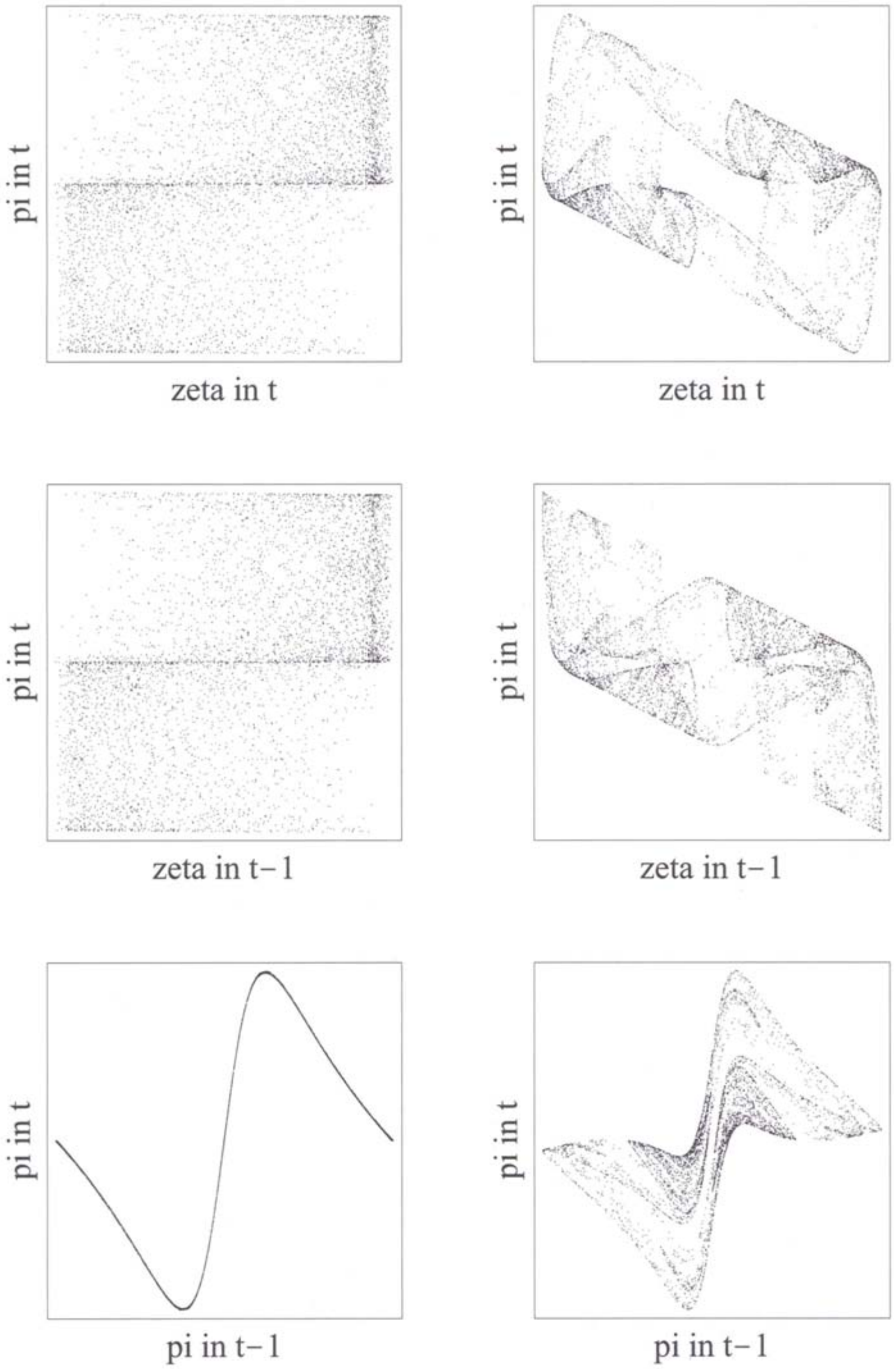

Figure 5: Emergence of strange attractors. In the panels from top to bottom, we plot π_t versus ζ_t, π_t versus ζ_{t-1}, and π_t versus π_{t-1}, respectively. The left-hand panels are based on the dynamics of the pitchfork scenario (top panels of figure 4) whereas the right-hand panels belong to the dynamics of the Neimark-Sacker scenario (bottom panels of figure 4).

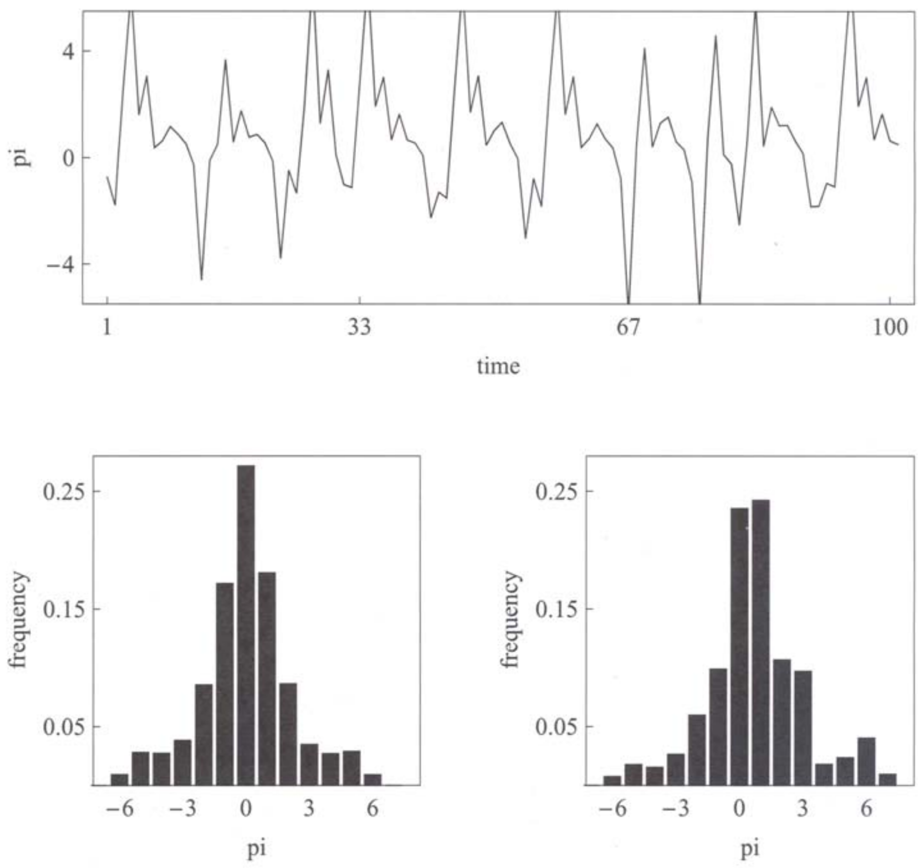

Figure 6: A piecewise linear supply function. The top panel shows the evolution of housing prices and the bottom right panel shows the corresponding distribution of these prices. Parameter setting as in the Neimark-Sacker scenario (with $b = 200$) but $P_{\mathbf{min}} = \overline{P} - 1$ and $\gamma = 0$. The bottom left panel shows the distribution of housing prices for the original model with the Neimark-Sacker parameter setting. In the right panel, the sample mean is 0.76, the median is 0.59 and the skewness is 0.13 whereas in the left panel, the mean is 0.00, the median is 0.00 and the skewness is -0.01.

Scenario	*c*	*d*	*e*	*f*	*g*	*h*
Flip	0.1	0.98	3.8	-	1	1
Pitchfork	0.6	0.98	0.0003	7.28	1	1
Neimark-Sacker	0.05	0.98	0.5	6	1	1

Table 1: Parameter settings for numerical results.